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c-Extremization from toric geometry

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Abstract

We derive a geometric formulation of the 2d central charge c_r from infinite families of 4d $\mathcal{N}=1$ superconformal field theories topologically twisted on constant curvature Riemann surfaces. They correspond to toric quiver gauge theories and are associated to D3 branes probing five dimensional Sasaki–Einstein geometries in the AdS/CFT correspondence. We show that c_r can be expressed in terms of the areas of the toric diagram describing the moduli space of the 4d theory, both for toric geometries with smooth and singular horizons. We also study the relation between a-maximization in 4d and c-extremization in 2d, giving further evidences of the mixing of the baryonic symmetries with the exact R-current in two dimensions.

1. Introduction

Anomalies play a crucial role in the analysis of conformal field theories. They provide consistency checks and impose several constraints on the behavior of RG flows and the existence of IR fixed points.

In four dimensions, a well studied anomaly is the coefficient of the Euler density of T_μ^μ, referred to as the central charge a. This quantity satisfies a c-theorem, decreasing from the UV

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to the IR endpoints of RG flows [1,2]. When considering $\mathcal{N} = 1$ superconformal field theories (SCFTs), the central charge a, non-perturbatively obtained in [3], is maximized by the exact R-current of the superconformal algebra [4]. The exact R-current turns out to be a linear combination of the trial UV R-current and the currents associated to the other global symmetries of the theory. By maximizing the central charge the mixing coefficients can be exactly computed. A peculiarity that features a-maximization in four dimensions is the absence of mixing of baryonic currents in the resulting expression for the exact R-current [5].

In two dimensions the trace anomaly is encoded in the central charge c, which satisfies a c-theorem as well [6]. As in four dimensions, in the case of 2d $\mathcal{N} = (0, 2)$ SCFTs the corresponding right central charge c_r is extremized by the exact 2d R-current that turns out to be a linear combination of a trial R-current and the currents corresponding to the other abelian global symmetries [7].

It is possible to construct classes of 2d SCFTs by partial topological twisting 4d SCFTs on Riemann surfaces Σ with constant curvature [8,9]. In order to preserve supersymmetry on the product space $\Sigma \times \mathbb{R}^{1,1}$, background magnetic fields for the global symmetries need to be turned on along Σ [10,11]. Starting from a 4d $\mathcal{N} = 1$ SCFT a specific choice of the fluxes allows to obtain $\mathcal{N} = (0, 2)$ supersymmetry in two dimensions. When the 4d theory has extended supersymmetry the resulting 2d SCFT can have enhanced supersymmetry [9,12–15]. A systematic classification of 2d SCFTs obtained by partial topological twist of $\mathcal{N} = 1, 2, 3, 4$ theories has been given in [16].

When considering 4d theories with an AdS_5 holographic dual description the topological twist can be reproduced at the gravitational level by turning on properly quantized fluxes for the (abelian) gauge symmetries in the bulk [17]. This triggers a RG flow across dimensions that, when restricting to the supergravity approximation, connects the original AdS_5 description to a warped $\mathrm{AdS}_3 \times \Sigma$ geometry. Alternatively, one can consider the full 10d geometry. Solving the BPS equations in this case should lead to a warped product $\mathrm{AdS}_3 \times \mathcal{M}_7$, where the general properties of the seven manifold $\mathcal{M}_7$ were originally discussed in [18,19]. This approach was taken in [20] for the infinite class of twisted Y^{pq} quiver gauge theories [21–23].

Under twisted compactification, the 2d central charge can be determined from the ’t Hooft anomalies of the 4d parent theory [20]. A comparison between the structure of the exact IR R-current in the 4d $\mathcal{N} = 1$ and in the 2d $\mathcal{N} = (0, 2)$ theories reveals that baryonic symmetries that do not mix with the 4d R-current can mix non-trivially with the 2d one [20].

An interesting class of 2d $\mathcal{N} = (0, 2)$ SCFTs is the one obtained by a partial topological twist of 4d $\mathcal{N} = 1$ toric quiver gauge theories that describe a stack of N D3 branes probing the tip of a toric Calabi–Yau threefold CY_3 over a 5d Sasaki–Einstein (SE) base X_5 with $U(1)^3$ isometry (see [24,25] and references therein).

The trial central charge of the 4d toric theory can be determined either geometrically, in terms of the geometrical data of the associated toric diagram [26,27] (see eq. (2.11)), or alternatively, through the holographic correspondence that provides a in terms of the X_5 volume parametrized by the Reeb vector of the dual supergravity solution [5,28–37] (see eq. (2.13)). The holographic dictionary translates a-maximization into the minimization of the X_5 volume [5,30].

For 2d SCFTs corresponding to twisted compactification of toric theories it would be nice to provide similar geometric and holographic prescriptions for computing c_r.

A correspondence between the 2d central charge c_r and the volume of the seven manifold $\mathcal{M}_7$ is currently lacking. A possible obstruction in finding a volume formula dual to c-extremization arises from the non-trivial mixing of baryonic currents in the 2d exact R-current. In fact, as a consequence, a putative volume formula for c_r should probably involve symmetries that are not necessarily isometries of the seven manifold, so making the generalization of the results in [30] to these cases not straightforward.

Leaving aside this open problem, in this paper we focus on the complementary problem of finding a geometric prescription to compute the central charge. We provide a general formula that expresses c_r at large N in terms of the toric data of the 4d parent theory. Having assigned the twisting generator $\mathcal{T}$ in four dimensions, our prescription is based on the non-trivial identification of the abelian fluxes on the compactifying Riemann surface with the $\mathcal{T}$-charges of the Perfect Matchings (PM) associated to the toric diagram of the 4d theory, and the mixing parameters in the 2d trial central charge with the PM R-charges. Assuming the validity of these identifications, the resulting formula for the trial central charge turns out to be very compact. Precisely, it reads

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| n_{\pi_I} \Delta_{\pi_J} \Delta_{\pi_K} \tag{1.1}$$

where the PM R-charges Δ_{π_J} and their $\mathcal{T}$-charges n_{π_I} satisfy constraints (2.8) and (2.23), respectively, as a legacy of the 4d conformality. The exact central charge is obtained by extremizing this expression as a function of Δ_{π_I}.

Result (1.1) fills a gap in the literature, representing the field theory dual of the holographic formula obtained by studying the $AdS_5 \to AdS_3$ flow in gauged supergravity, in the presence of a generic amount of vector multiplets [15,38–40].

We provide several checks of our proposal by applying prescription (1.1) to 2d SCFTs for which the field content and the corresponding charges are known as functions of the mixing parameters and the fluxes on Σ. In all the cases the central charge coincides with the large N expression computed using the prescription in [7]. Moreover, we extend our prescription to the case of twisted compactification of 4d toric theories with singular horizons.

As already mentioned, in four dimensions the exact R-symmetry of toric quiver gauge theories is a mixture of the $U(1)^3$ symmetries of X_5, whereas the baryonic ones, corresponding to the non-anomalous combination of the $U(1) \subset U(N)$ gauge groups, decouple in the IR and do not play any role. When flowing to two dimensions, baryonic symmetries can mix with the exact R-current as explicitly shown in the particular case of $X_5 = Y^{pq}$ for which there is a single baryonic symmetry [20]. By studying several examples of increasing complexity, we show that this picture is general and holds for models with a larger amount of baryonic symmetries.

The paper is organized as follows. In section 2 we review some basic aspects of toric quiver gauge theories and the topological twist, necessary to our analysis. In section 3 we derive the expression of c_r in terms of the toric data of the 4d theory. We first study cases with smooth horizons, correctly reproducing the behavior of c_r as a function of the R-charges. We confirm the validity of this formula by studying many examples of increasing complexity. In section 4 we consider the case of non-smooth horizons, describing the prescription for obtaining c_r in terms of the toric data of the 4d theory. In section 5 we study the compactification of del Pezzo gauge theories, dP_2 and dP_3, with respectively two and three non-anomalous baryonic symmetries, showing their mixing in the exact 2d R-current. Then we study a case with a generic amount of baryonic symmetries, by showing the mechanism in necklace quivers, denoted as L^{pqp} theories. In section 6 we discuss the interpretation and possible implications of our results. For 2d theories obtained by topologically twisted reduction of dP_2 and dP_3 toric theories, in appendix A we report the explicit values for the parameters of $U(1)$ mixing for particular choices of the fluxes.

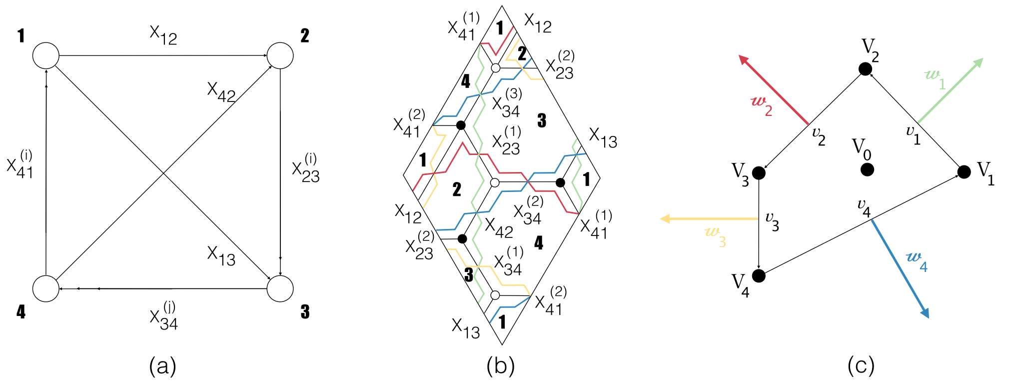


Fig. 1. Quiver, dimer and toric diagram of the dP_1 model. In quiver (a) the number of arrows on the straight lines indicates the number of fields connecting two nodes. In panel (c) primitive normal vectors w_I of the toric diagram are also indicated and the different colors clarify their relation with the zig-zag paths in the dimer, panel (b). This is useful for reading the R-charges of the fields in terms of the charges of the zig-zag paths or of the perfect matchings.

2. Review

We start our discussion by reviewing the main aspects of toric quiver gauge theories and their twisted compactification on constant curvature Riemann surfaces.

2.1. Toric quiver gauge theories

Toric quiver gauge theories [41] describe the near horizon limit of a stack of N D3 branes probing the tip of a CY_3 cone over a 5d SE X_5, characterized by a $U(1)^3$ action on the metric. The dual $\mathcal{N}=1$ SCFTs are described by quiver gauge theories whose nodes carry $U(N)$ gauge factors and are connected by oriented arrows, representing bifundamental matter fields.

In order to exemplify the discussion we consider the explicit case of a gauge theory living on a stack of N D3 branes probing the first del Pezzo singularity, dP_1. It has four gauge groups and the corresponding quiver is represented in Fig. 1(a). The superpotential

$$W = -\epsilon_{\alpha\beta} X_{12} X_{23}^{(\alpha)} X_{34}^{(3)} X_{41}^{(\beta)} + \epsilon_{\alpha\beta} X_{23}^{(\alpha)} X_{34}^{(\beta)} X_{42} + \epsilon_{\alpha\beta} X_{13} X_{34}^{(\alpha)} X_{41}^{(\beta)} \tag{2.1}$$

is subject to the toric condition, which requires that each field appears in exactly two terms having opposite signs.[1] This model has a $SU(2)\times U(1)$ flavor symmetry that, together with the $U(1)_R$ R-symmetry, builds up the isometry group of dP_1. In general, there are also baryonic symmetries associated to the non-trivial second cohomology group of X_5. These symmetries can be obtained from the $U(1)\subset U(N)$ gauge factors. They are IR free and at low energies decouple from the dynamics, becoming global symmetries. In quivers with a chiral-like matter content as the ones considered here, some of these $U(1)$'s are anomalous. The non-anomalous abelian factors correspond to the aforementioned baryonic symmetries. For the specific example of dP_1, to begin with there are four $U(1)_i \subset U(N)_i$ global symmetries of baryonic type with $T_{i=1,\ldots,4}$ generators. Two combinations are anomalous and one decouples. We are then left with just a single non-anomalous baryonic symmetry that can be for example identified with the combination $2T_1 - T_2 + T_3$.

[1] For an exhaustive review on toric gauge theories we refer to [24,25].

When flowing to the IR fixed point abelian flavor symmetries can mix with the R-current to form the exact R-symmetry, whereas the baryonic symmetries do not mix, as discussed in [5,42]. This is a general feature of this family of 4d SCFTs.

For a quiver theory with n_G gauge groups the mixing coefficients of global symmetries into the exact R-symmetries are obtained by maximizing the central charge [4]

$$a_{FT} = \frac{3}{32}(3\,\mathrm{Tr}R^3 - \mathrm{Tr}R) = \frac{3}{32}\left[2n_G(N^2-1) + \sum_{i=1}^{n_F} \dim(\rho_i)(3(R_i-1)^3 - (R_i-1))\right] \tag{2.2}$$

where the first term is the contribution of the gaugini, n_F it the total amount of matter multiplets, $\dim(\rho_i)$ is the dimension of the corresponding representation and R_i the R-charge of the scalar component of the i-th multiplet. For matter multiplets in the bifundamental and/or adjoint representations, at large N the central charge is further simplified by the constraint $\mathrm{Tr}R = 0$ and we read

$$a_{FT} = \frac{9}{32}N^2\left(n_G + \sum_{i=1}^{n_F}(R_i-1)^3\right) + \mathcal{O}(1) \tag{2.3}$$

For toric gauge theories the R_i charges can be determined directly from the geometric data of the singularity [5,26–37], as we now review.

First of all, we recollect how to construct the toric diagram corresponding to a given quiver gauge theory. One embeds the quiver diagram (for the dP_1 case see Fig. 1(a)) in a two dimensional torus. The resulting planar diagram can be dualized by inverting the role of faces and nodes, thus obtaining a bipartite diagram, called dimer, where faces correspond to gauge groups, edges to fields and nodes to superpotential interactions (for the dP_1 model it is given in Fig. 1(b)). The toric condition of the superpotential translates into a bipartite structure of the dimer. From the dimer one can construct perfect matchings (PM's), that is collections of edges (fields) characterized by the property that each node is connected to one and only one edge of the set.

One can introduce a new set of formal variables π_I associated to each PM. These variables are defined by the relations

$$X_{ij} \equiv \prod_I (\pi_I)^{M_I(X_{ij})} \tag{2.4}$$

where the product is taken over all the PMs and

$$M_I(X_{ij}) = \begin{cases} 0 & \text{if } X_{ij} \text{ does not belong to the set of } \pi_I \\ 1 & \text{if } X_{ij} \text{ belongs to the set of } \pi_I \end{cases} \tag{2.5}$$

The π_I's provide a convenient set of variables that can be used to parametrize the abelian moduli space of the quiver gauge theory. The advantage of using the PMs variables π_I instead of the more natural set of scalar components of the chiral fields X_{ij} comes from the fact that using definition (2.4) the F-term equations are trivially satisfied. This is a consequence of the fact that, in this basis, each term in the superpotential becomes equal to $\pm\prod_I \pi_I$, with I ranging over all PMs.

To each PM we can associate a signed intersection number, ± 1 or 0, with respect to a basis of 1-cycles of the first homology of the torus. The signs can be inferred from the bipartite structure of the dimer. For each PM these two intersection numbers are the first two coordinates of 3d vectors $V_I \equiv (\cdot, \cdot, 1)$ defining a convex integral polygon, named toric diagram, embedded into a 2d section of a 3d lattice at height one.

In the dP_1 case, the toric diagram is given by the 3d vectors V_I that are associated to the PM's as follows

PM	primitive vector	PM	primitive vector
$\pi_1 = \{X_{34}^{(3)}, X_{42}, X_{13}\}$	$V_1 = (1, 0, 1)$	$\pi_5 = \{X_{12}, X_{13}, X_{42}\}$	$V_0 = (0, 0, 1)$
$\pi_2 = \{X_{23}^{(1)}, X_{34}^{(1)}, X_{41}^{(1)}\}$	$V_2 = (0, 1, 1)$	$\pi_6 = \{X_{13}, X_{23}^{(1)}, X_{23}^{(2)}\}$	$V_0 = (0, 0, 1)$
$\pi_3 = \{X_{12}, X_{34}^{(1)}, X_{34}^{(2)}\}$	$V_3 = (-1, 0, 1)$	$\pi_7 = \{X_{34}^{(1)}, X_{34}^{(2)}, X_{34}^{(3)}\}$	$V_0 = (0, 0, 1)$
$\pi_4 = \{X_{23}^{(2)}, X_{34}^{(2)}, X_{41}^{(2)}\}$	$V_4 = (-1, -1, 1)$	$\pi_8 = \{X_{41}^{(1)}, X_{41}^{(2)}, X_{42}\}$	$V_0 = (0, 0, 1)$

(2.6)

and the corresponding toric diagram is drawn in Fig. 1(c).

It is also useful to review the notion of zig-zag paths. Given the set of primitive vectors V_I, one can define primitive normal vectors w_I, orthogonal to the edges of the toric diagram, $v_I \equiv (V_{I+1} - V_I)$, $I = 1, .., 4$ with $V_5 \equiv V_1$ (see Fig. 1(c)). These vectors are in 1–1 correspondence with a set of paths, made out of edges of the dimer, called zig-zag paths and represented in Fig. 1(b). They are oriented closed loops on the dimer that turn maximally left (right) at the black (white) nodes. The zig-zag paths correspond to differences of consecutive PM's lying at the corners of the toric diagram (and, if present, on the perimeter) and are associated to the $U(1)$ global symmetries of the superpotential.

Vice versa, given a particular toric diagram with d external points, it is possible to identify the main features of the corresponding quiver gauge theory as follows:

- The number of $U(N)$ gauge groups describing the quiver is given by twice the area of the toric diagram.
- The matter content of the theory (type of bifundamental fields and their degeneracy) can be inferred from the edges v_I of the toric diagram [43], up to Seiberg duality, or equivalently toric phases, corresponding to Yang–Baxter transformations on the zig zag paths [44]. In its minimal toric phase a set Φ_{IJ} of bifundamental fields X_{ij} is assigned to each pair $(I, J)_{I,J=1,..,d}$ with degeneracy $|\det(v_I, v_J)|$.
- The corresponding spectrum of R_{IJ} charges is determined by assigning a R-charge Δ_{π_I} to each PM on the boundary of the toric diagram and using the following prescription [33]
$$\begin{cases} R_{IJ} = \sum_{K=I+1}^{J} \Delta_{\pi_K} & I < J \\ R_{IJ} = 2 - \sum_{K=J+1}^{I} \Delta_{\pi_K} & I > J \end{cases} \tag{2.7}$$
where the charges Δ_{π_I} are subject to the constraint
$$\sum_{I=1}^{d} \Delta_{\pi_I} = 2 \tag{2.8}$$
to ensure that each superpotential term has R-charge equal to two. A geometric interpretation of (2.8) can be given in terms of the *isoradial embedding* [44].
The fact that the degeneracy of the fields with a given R-charge R_{IJ} is given by $|\det(v_I, v_J)|$ has a nice geometric interpretation in terms of zig-zag paths [33].
- The number d of external vertices of the toric diagram also determines the total number of non-anomalous $U(1)$ global symmetries of the gauge theory, which are identified as one R-symmetry, two flavor symmetries and $(d-3)$ baryonic symmetries. Analogously to (2.8), the

condition for the superpotential to be neutral with respect to any non-R symmetry translates into

$$\sum_{I=1}^{d} Q^{\mathbf{J}}_{\pi_I} = 0 \qquad \mathbf{J} = 1, \dots, d-1 \tag{2.9}$$

where $Q^{\mathbf{J}}_{\pi_I}$ is the charge of the I-th PM with respect to the $\mathbf{J}$-th symmetry.

- From the geometric data one can also identify the anomalies of the theory. The crucial observation is that the areas of the triangles of the toric diagram are related to the coefficients of the 't Hooft anomalies of the field theory as [26,27]

$$\mathrm{Tr}_{4\mathrm{d}}(\mathcal{T}_I \mathcal{T}_J \mathcal{T}_K) = \frac{N^2}{2} |\det(V_I, V_J, V_K)| \tag{2.10}$$

where $\mathcal{T}_I$ are global symmetry generators and the trace $\mathrm{Tr}_{4\mathrm{d}}$ is taken over the 4d fermions with the insertion of the 4d chirality operator.

Consequently, the central charge can be written as [27]

$$a_{geom} \equiv \frac{9}{64} N^2 |\det(V_I, V_J, V_K)| \, \Delta_{\pi_I} \Delta_{\pi_J} \Delta_{\pi_K} \tag{2.11}$$

This expression is equivalent to (2.3) once we take into account the mapping between the two sets of R_i and Δ_{π_I} charges, eq. (2.7).

In the case of the dP_1 model, using definition (2.4) we find the following map between the sets of fields Φ_{IJ} obtained from the toric diagram and the fields given by the quiver description

(I, J)	$\lvert\det(v_I, v_J)\rvert$	Φ_{IJ}	R_{IJ}
$(4, 1)$	3	$\{X_{13}, X_{34}^{(3)}, X_{42}\}$	Δ_{π_1}
$(1, 2)$	2	$\{X_{23}^{(1)}, X_{41}^{(1)}\}$	Δ_{π_2}
$(2, 3)$	1	$\{X_{12}\}$	Δ_{π_3}
$(3, 4)$	2	$\{X_{23}^{(2)}, X_{41}^{(2)}\}$	Δ_{π_4}
$(1, 3)$	1	$\{X_{34}^{(1)}\}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$
$(2, 4)$	1	$\{X_{34}^{(2)}\}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$

(2.12)

where Δ_{π_I} satisfy (2.8) and all internal PMs are assigned zero charge under all $U(1)$ symmetries.

The example we have considered has a smooth horizon where all the external points of the toric diagram correspond to corners. In this case the prescription for assigning R-charge to the bifundamental fields is unambiguously given in (2.7). In the case of singular horizons there are also points on the perimeter of the toric diagram that do not correspond to corners. These points have a degeneracy (given by a binomial coefficient), as they correspond to more than one PM. The assignment of the R-charges in terms of the external PM's may then become ambiguous. According to the prescription in [33,34], at these points one sets to zero the R-charges of the PM's that do not determine any zig-zag path, being then left with an unambiguous assignment of Δ_{π_I} charges.

The holographic correspondence provides the following relation between the central charge and the X_5 volume [28]

$$a_{\mathrm{holo}} \equiv \frac{N^2 \pi^3}{4 \, \mathrm{vol}(\mathrm{X}_5(\mathbf{b}))} \tag{2.13}$$

where the volume is parameterized in terms of the components of the Reeb vector $\mathbf{b}$, a constant norm Killing vector that commutes with the X_5 isometries. It follows that the a-maximization prescription that determines the exact R-current in field theory corresponds to the volume minimization in the gravity dual.

When the cone over X_5 is toric the central charge can be directly obtained from the toric geometry. In fact, the X_5 volume can be expressed as [30]

$$\mathrm{vol}(X_5) = \frac{\pi}{6} \sum_{I=1}^{d} \mathrm{vol}(\Sigma_I) \tag{2.14}$$

where d represents the number of vertices and $\mathrm{vol}(\Sigma_I)$ corresponds to the volume of a 3-cycle Σ_I, on which D3 branes, corresponding to dibaryons, are wrapped [45].

Holographic data also determine the Δ_{π_I} charges that can be parameterized in terms of the components of the Reeb vector $\mathbf{b}$. Using the explicit parameterization [29]

$$\Delta_{\pi_I}(\mathbf{b}) = \frac{\pi}{3} \frac{\mathrm{vol}(\Sigma_I(\mathbf{b}))}{\mathrm{vol}(X_5(\mathbf{b}))} \tag{2.15}$$

it is easy to show the equivalence between a_{geom} in eq. (2.11) and a_{holo} in eq. (2.13).

2.2. Twisted compactification

In this section we review the main aspects of partial topologically twisted compactifications of a 4d $\mathcal{N}=1$ SCFTs on a genus g Riemann surface Σ, and the computation of the central charge c_r for the corresponding 2d SCFTs, directly from 4d anomaly data.

When placing a 4d $\mathcal{N}=1$ SCFT on $\Sigma \times \mathbb{R}^{1,1}$, supersymmetry is generally broken by the coupling with the Σ curvature. In order to (partially) preserve it one performs a twist [8] by turning on background gauge fields along Σ for an abelian 4d R-symmetry t_R that assigns integer charges to the fields. Choosing its flux to be proportional to the curvature, its contribution to the Killing spinor equations cancels the contribution from the spin connection and possibly non-trivial solutions for Killing spinors can be found. More generally, one can also turn on properly quantized background fluxes along the Σ directions for other abelian global symmetries. In this case supersymmetry is preserved if the associated gaugino variations vanish as well. Summarizing, the most general twist is performed along the generator

$$T = \kappa\, T_R + \sum_{\mathbf{I}=1}^{n_A} b_{\mathbf{I}} T_{\mathbf{I}} \tag{2.16}$$

where $\kappa = 0$ for the torus and $\kappa = \pm 1$ for curved Riemann surfaces.[2] Here n_A refers to the number of abelian $T_{\mathbf{I}}$ generators of non-R global symmetries (both flavor and baryonic ones) and $b_{\mathbf{I}}$ are the corresponding background fluxes. For generic choices of the fluxes $\mathcal{N} = (0, 2)$ supersymmetry is preserved on $\mathbb{R}^{1,1}$ [16].

After the twist the trial 2d R-symmetry generator is a mixture of the abelian generator T_R and the other $T_{\mathbf{I}}$ generators

$$R = T_R + \sum_{\mathbf{I}=1}^{n_A} \epsilon_{\mathbf{I}} T_{\mathbf{I}} \tag{2.17}$$

[2] We use conventions of [46] that differ by a factor 2 from the conventions previously used in [16].

where $\epsilon_{\mathbf{I}}$ are the mixing coefficients and $T_R, T_{\mathbf{I}}$ are meant to act on fields reorganized in 2d representations.

At the IR fixed point the $\epsilon_{\mathbf{I}}$ coefficients have to extremize the 2d central charge $c_r = 3k_{RR} \equiv 3\mathrm{Tr}_{2\mathrm{d}}(RR)$ [15]. Practically, the relevant anomaly coefficient k_{RR} can be obtained from the anomaly polynomial I_6 expressed in terms of the triangular anomalies of the 4d theory, by integrating I_6 on Σ and matching the resulting expression with the general structure of the anomaly polynomial I_4 in two dimensions [20]. In particular, one obtains the trial central charge

$$\begin{aligned} c_r &= 3\,\eta_\Sigma \mathrm{Tr}_{4\mathrm{d}}(TR^2) \\ &= 3\,\eta_\Sigma \left[(b_{\mathbf{K}} k_{\mathbf{KIJ}} + \kappa k_{R\mathbf{IJ}})\epsilon_{\mathbf{I}}\epsilon_{\mathbf{J}} + 2(b_{\mathbf{K}} k_{R\mathbf{KI}} + \kappa k_{RR\mathbf{I}})\epsilon_{\mathbf{I}} + (b_{\mathbf{K}} k_{RR\mathbf{K}} + \kappa k_{RRR})\right] \end{aligned} \tag{2.18}$$

where $\eta_\Sigma = 2|g-1|$ for $g \neq 1$ and $\eta_\Sigma = 1$ for $g = 1$, and we have defined $k_{\mathbf{KIJ}} \equiv \mathrm{Tr}_{4\mathrm{d}}(T_{\mathbf{K}} T_{\mathbf{I}} T_{\mathbf{J}})$, $k_{R\mathbf{IJ}} \equiv \mathrm{Tr}_{4\mathrm{d}}(RT_{\mathbf{I}}T_{\mathbf{J}})$ and similarly for the other trace coefficients.

The exact central charge is finally obtained by extremizing with respect to the variables $\epsilon_{\mathbf{I}}$ [7]. At the fixed point we have

$$\epsilon^*_{\mathbf{I}} = -\eta_\Sigma k^{-1}_{\mathbf{IJ}}\,(k_{R\mathbf{JK}} b_{\mathbf{K}} + \kappa k_{RR\mathbf{J}}) \tag{2.19}$$

where

$$k_{\mathbf{IJ}} = \eta_\Sigma\,(\kappa k_{R\mathbf{IJ}} + b_{\mathbf{K}} k_{\mathbf{KIJ}}) \tag{2.20}$$

From (2.19) we observe that coefficients $\epsilon^*_{\mathbf{I}}$ are generically non-vanishing for any choice of the $b_{\mathbf{I}}$ fluxes. In particular, this is true for the coefficients associated to baryonic symmetries, which then do mix with the exact R-current in two dimensions, even if they do not in the original 4d theory. This pattern has been already observed in [20] for the Y^{pq} family. In section 5 we will study toric quiver gauge theories with a larger amount of baryonic symmetries, confirming that they generically mix with the 2d exact R-current after the twisted compactification on Σ.

Starting from a 4d toric theory with n_G $U(N)$ gauge groups and n_F massless chiral fermions, to each 2d field surviving the compactification on Σ we can associate a T-charge n_i and a R-charge R_i according to (see eqs. (2.16) and (2.17))

$$n_i = \kappa\, r_i + Q_i^{\mathbf{J}} b_{\mathbf{J}}, \quad R_i = r_i + Q_i^{\mathbf{J}} \epsilon_{\mathbf{J}} \qquad i = 1, \ldots, n_F \tag{2.21}$$

where r_i is the R-charge respect to the 4d R-current and $Q_i^{\mathbf{J}}$ is the charge matrix of the fermions respect to the global $U(1)$ non-R symmetries, inherited from the 4d parent fields. Therefore, applying prescription (2.18) we find that at large N the central charge before extremization is given by

$$c_r = 3N^2 \eta_\Sigma \left(\kappa\, n_G + \sum_{i=1}^{n_F} (n_i - \kappa)(R_i - 1)^2\right) + \mathcal{O}(1) \tag{2.22}$$

This formula is general and applies to any 2d SCFT obtained from compactification of a 4d quiver gauge theory on a Riemann surface with curvature κ. Through R_i it depends parametrically on the mixing coefficients $\epsilon_{\mathbf{I}}$ that need to be determined by the 2d extremization procedure.

As reviewed in the previous sub-section, in the case of 4d toric quiver theories we can parametrize the $U(1)$ charges in terms of PM variables Δ_{π_I} and $Q^{\mathbf{J}}_{\pi_I}$. When twisting, we can also assign to PMs a further n_{π_I} charge with respect to the twisting T symmetry (2.16) as (in this case $n_A = d - 1$)

$$n_{\pi_I} = \kappa \Delta_{\pi_I} + b_{\mathbf{J}} Q^{\mathbf{J}}_{\pi_I} \qquad \text{with} \qquad \sum_{I=1}^{d} n_{\pi_I} = 2\kappa \tag{2.23}$$

where the constraint on n_{π_I} follows from (2.8) and (2.9).

The r_i and $Q_i^{\mathbf{J}}$ charge assignments in two dimensions, eq. (2.21), need necessarily to respect the original constraints arising from the condition of superconformal invariance for the 4d superpotential. In particular, given the superpotential $W = \sum_\alpha W_\alpha$, these constraints imply that for each superpotential term W_α the conditions $\sum_{i\in W_\alpha} r_i = 2$ and $\sum_{i\in W_\alpha} Q_i^{\mathbf{J}} = 0$ hold. Consequently, from (2.21) we read

$$\sum_{i\in W_\alpha} R_i = 2\,, \qquad \sum_{i\in W_\alpha} n_i = 2\kappa \tag{2.24}$$

Now, we can think of the dimensional flow from the original 4d theory to the resulting 2d one as being accompanied by the set of toric data $(\Delta_{\pi_I}, Q^{\mathbf{J}}_{\pi_I}, n_{\pi_I})$ that parametrize the $U(1)$ charges in 4d and, consequently, that can still be used to parametrize the corresponding charges in two dimensions. Using this parametrization reinterpreted as charge parametrization for 2d fields, constraints (2.24) are traded with (2.8), (2.9) and (2.23).

3. c_r from toric geometry

For the class of 2d SCFTs obtained from the topologically twist reduction of toric quiver gauge theories, we now provide a general prescription for determining the central charge c_r directly in terms of the geometry of the toric diagram associated to the original 4d parent theory. This is the main result of the paper, which we are going to check in the successive sub-sections for a number of explicit examples.

3.1. Reading the 2d central charge from the toric diagram

To this end, we consider a toric gauge theory twisted along the abelian generator

$$T = \sum_{I=1}^{d} a_I \mathcal{T}_I \qquad \text{with} \qquad \sum_{I=1}^{d} a_I = 2\kappa \tag{3.1}$$

where I runs over the d external points of the toric diagram. To be consistent with the conventions used so far, the abelian $\mathcal{T}_I$ generators are chosen so that they assign charge one to the superpotential of the 4d theory. It is always possible to construct such a set of generators by combining the generators of the 4d trial R-current, the two flavor symmetries and the $(d-3)$ non-anomalous baryonic symmetries that appear in (2.16). The new fluxes are subject to the constraint in (3.1) in order to ensure $\mathcal{N} = (0, 2)$ supersymmetry in 2d. They need to be further constrained in such a way that each flux $b_{\mathbf{I}}$ in (2.16) is properly quantized.

Accordingly, the 2d trial R-symmetry can be written as

$$R = \sum_{I=1}^{d} \epsilon_I \mathcal{T}_I \qquad \text{with} \qquad \sum_{I=1}^{d} \epsilon_I = 2 \tag{3.2}$$

where the constraint follows from the requirement for R to be a canonical normalized R-current.

The 2d central charge c_r expressed in terms of the 4d anomaly coefficients $\mathrm{Tr}_{4d}(\mathcal{T}_I\mathcal{T}_J\mathcal{T}_K)$, the a_I fluxes and the mixing parameters ϵ_I becomes (see eq. (2.18))

$$c_r = 3\,\eta_\Sigma \mathrm{Tr}_{4\mathrm{d}}(T R^2) = 3\,\eta_\Sigma \mathrm{Tr}_{4\mathrm{d}}(\mathcal{T}_I \mathcal{T}_J \mathcal{T}_K)\, a_I \epsilon_J \epsilon_K \tag{3.3}$$

In the case of toric theories the anomaly coefficients are given by (2.10) in terms of the areas of the triangles of the toric diagram. Therefore, the 2d central charge can be rewritten as

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| a_I \epsilon_J \epsilon_K \tag{3.4}$$

In order to complete the map between the 2d field theory and the 4d geometric data we need to find a prescription for parameterizing the a_I fluxes and the mixing parameters ϵ_I in terms of the PM's associated to the external vertices of the toric diagram. To this end, we observe that the constraints satisfied by a_I and ϵ_I, eqs. (3.1), (3.2), are the same as the constraints satisfied by Δ_{π_I}, eq. (2.8) and n_{π_I}, eq. (2.23), and we are naturally led to identify $\epsilon_I \equiv \Delta_{\pi_I}$ and $a_I \equiv n_{\pi_I}$. Therefore, in the large N limit the central charge c_r for the 2d SCFT obtained from a 4d toric quiver gauge theory topologically twisted on a 2d constant curvature Riemann surface can be expressed entirely in terms of the toric data by the formula

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| n_{\pi_I} \Delta_{\pi_J} \Delta_{\pi_K} \tag{3.5}$$

with Δ_{π_J} and n_{π_I} satisfying constraint (2.8) and (2.23). The exact central charge for the 2d SCFT is then obtained by extremizing (3.5) as a function of Δ_{π_I}.

We note that equation (3.5) gives also the left central charge c_l. In fact, in the large N limit the gravitational anomaly $k = c_r - c_l$ is vanishing, being a linear combination of $\mathrm{Tr}_{4d} R$ and $\mathrm{Tr}_{4d} \mathcal{T}_\mathbf{I}$ that for toric theories are subleading in N [47] (whereas the traces of the baryonic symmetries vanish also at finite N [5]).

Our proposal (3.5) requires some direct check on explicit examples that we report below. However, a holographical confirmation can be already found in the analysis of the $\mathrm{AdS}_5 \to \mathrm{AdS}_3$ flow engineered in gauged supergravity [17]. In this case we need to consider a consistent truncation of $\mathrm{AdS}_5 \times X_5$, a 5d theory with a gravity multiplet, n_V vector multiplets and n_H hypermultiplets. The graviphoton plays the role of the R-symmetry current, while the n_V vector multiplets correspond to the non-R global currents of the holographic dual field theory that remain as massless vector multiplets in a given truncation. In general $n_V \leq n_A$. The hypermultiplets impose constraints that correspond to the vanishing of the R-current anomaly in the dual field theory [32]. When flowing to AdS_3 and using the Brown–Henneaux formula [48] in this setup, it was observed [15,38,39] that c_r can be expressed in terms of R-charges $\hat{r}^I$ and fluxes $\hat{a}^J$ as

$$c_r = \frac{2\pi^3 N^2 \eta_\Sigma}{3\mathrm{vol}(\mathrm{X}_5)} C_{IJK} \hat{a}^I \hat{r}^J \hat{r}^K \tag{3.6}$$

where the constraints $\sum \hat{r}^I = 2$ and $\sum \hat{a}^I = 2\kappa$ need to be imposed. In this formula C_{IJK} are the Chern–Simons coefficients of the dual supergravity, the R-charges $\hat{r}^I$ are obtained from the sections of the special geometry corresponding to the (constrained) scalars in the vector multiplets, and the prepotentials of $\mathcal{N} = 2$ AdS_5 gauged supergravity. The constants $\hat{a}^I$ are the coefficients of the volume forms in the reduction of the 5d vector multiplets to 3d.

On the other hand, the C_{IJK} coefficients are the holographic duals of the cubic 't Hooft anomaly coefficients, which for toric quiver gauge theories correspond to the areas of the triangles in the toric diagrams, eq. (2.10). Therefore

$$C_{IJK} = \frac{N^2}{2} |\det(V_I, V_J, V_K)| \tag{3.7}$$

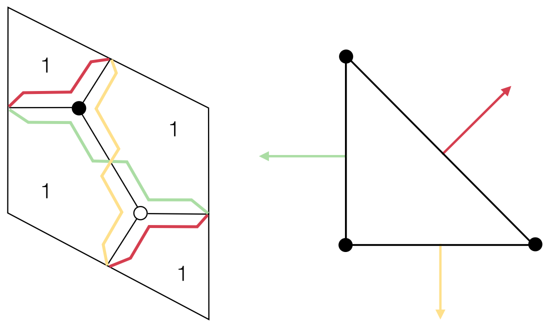


Fig. 2. Dimer, zig-zag paths and toric diagram of S^5.

If we naturally identify the R-charges $\hat{r}^I$ with the Δ_{π_I} charges assigned to the PM's, and similarly the $\hat{a}_I$ fluxes with the set of n_I fluxes (they satisfy the same constraints $\sum \Delta_{\pi_I} = 2$ and $\sum n_{\pi_I} = 2\kappa$) we obtain our proposal (3.5).

3.2. Examples

In the remaining part of this section we test formula (3.5) on examples of increasing complexity. As a warm-up we consider the cases of $X_5 = S^5$ corresponding to $\mathcal{N} = 4$ SYM and $X_5 = T^{1,1}$ corresponding to the conifold [49]. Then we move to two more complicated cases, namely the second and third del Pezzo surfaces. We conclude the analysis by considering infinite families of quiver gauge theories associated to the Y^{pq} [21–23], L^{pqr} [50,51] and X^{pq} [52] geometries.

The strategy is the following. For each 4d model we use the general formula (2.22) to compute the central charge of the corresponding 2d SCFT obtained after twisted compactification. Then, we determine the parametrization of the R-charges and fluxes in terms of the toric data according to our prescription in section 3.1. Finally, we check that using this parametrization in (2.22) we obtain the central charge as given by (3.5).

$\mathcal{N} = 4$ SYM

The first example that we consider corresponds to the case of $X_5 = S^5$. In this case the dual gauge theory is $\mathcal{N} = 4$ SYM and its twisted compactification on a Riemann surface has been discussed in [7,9,53]. The 4d field theory can be studied as a toric quiver gauge theory in $\mathcal{N} = 1$ language. In this formulation the global symmetry corresponds to the $U(1)^3$ abelian subgroup of $SO(6)_R$. The quiver has a single node with three adjoint superfields Φ_i and superpotential

$$W = \Phi_1[\Phi_2, \Phi_3] \tag{3.8}$$

The dimer, the zig-zag paths and the toric diagram are shown in Fig. 2.

By reducing this theory on Σ the topological twist is performed along the $U(1)^3$ subgroup of the $SO(6)_R$. This corresponds to turning on three fluxes, one for each $U(1)$ factor, constraining their sum to be equal to the curvature κ. From the general expression (2.22) we can read the 2d central charge at large N

$$c_r = 3N^2\eta_\Sigma\left(\kappa + \sum_{i=1}^{3}(n_{\Phi_i} - \kappa)(R_{\Phi_i} - 1)^2\right) \tag{3.9}$$

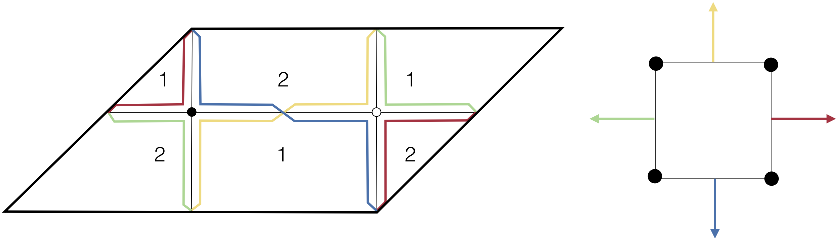


Fig. 3. Dimer, zig-zag paths and toric diagram of $T^{1,1}$.

where R_{Φ_i} are the R-charges and n_{Φ_i} the associated fluxes of the three adjoint fields. These variables are constrained by the relations $R_{\Phi_1} + R_{\Phi_2} + R_{\Phi_3} = 2$ and $n_{\Phi_1} + n_{\Phi_2} + n_{\Phi_3} = 2\kappa$.

Alternatively, we can compute the 2d central charge from (3.5) and find

$$c_r = 3N^2 \eta_\Sigma \big(n_{\pi_1} \Delta_{\pi_2} \Delta_{\pi_3} + n_{\pi_2} \Delta_{\pi_3} \Delta_{\pi_1} + n_{\pi_3} \Delta_{\pi_1} \Delta_{\pi_2} \big) \tag{3.10}$$

In order to check this result against (3.9) we need to express R-charges and fluxes in terms of the ones of the PM's. This can be done with the prescription discussed in section 2.1. The three zig-zag paths in Fig. 2 are the three possible combinations of two adjoints, $\Phi_i \Phi_j$. It follows that each adjoint field corresponds to the intersection of two primitive normal vectors w_I of the toric diagram. Furthermore in this case each external PM corresponds to one of the adjoint fields. Therefore the charge and the flux assigned to each field correspond to the charge and the flux assigned to each external PM

$$\begin{aligned} R_{\Phi_1} &= \Delta_{\pi_1}, \quad & R_{\Phi_2} &= \Delta_{\pi_2}, \quad & R_{\Phi_3} &= \Delta_{\pi_3} \\ n_{\Phi_1} &= n_{\pi_1}, \quad & n_{\Phi_2} &= n_{\pi_2}, \quad & n_{\Phi_3} &= n_{\pi_3} \end{aligned} \tag{3.11}$$

By substituting this parameterization in (3.9) we can easily prove that in this case the central charge is equivalent to (3.10) if constraints (2.8) and (2.23) are imposed.

The conifold

As a second example we study the case of the conifold, corresponding to $X_5 = T^{1,1}$. The model consists of a $SU(N) \times SU(N)$ gauge theory with two pairs of bifundamental a_i and anti-bifundamental b_i fields connecting the gauge groups and interacting through the superpotential

$$W = \epsilon_{ij} \epsilon_{lk} a_i b_l a_j b_k \tag{3.12}$$

The dimer, the zig-zag paths and the toric diagram are shown in Fig. 3. In this case the flavor symmetry is $SU(2)^2$ and one baryonic $U(1)$ symmetry is also present. The R-charges of the four fields, R_{a_i} and R_{b_i} are constrained by $R_{a_1} + R_{a_2} + R_{b_1} + R_{b_2} = 2$.

When twisting the theory on Σ we introduce T-fluxes defined in (2.21). In this case they are $n_{a_1}, n_{a_2}, n_{b_1}$ and n_{b_2}, constrained by $n_{a_1} + n_{a_2} + n_{b_1} + n_{b_2} = 2\kappa$.

The 2d central charge can be written at large N, using eq. (2.22)

$$c_r = 3N^2 \eta_\Sigma \left[2\kappa + \sum_{i=1}^{2} \left((n_{a_i} - \kappa)(R_{a_i} - 1)^2 + (n_{b_i} - \kappa)(R_{b_i} - 1)^2 \right) \right] \tag{3.13}$$

This formula can be reproduced from the geometry of the toric diagram using prescription (3.5). To prove it, we start by ordering the vectors V_I in the toric diagram as

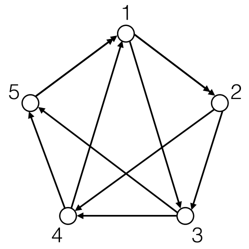


Fig. 4. Quiver of the $dP_2^{(I)}$ model.

$$V_1 = (0, 0, 1), \quad V_2 = (1, 0, 1), \quad V_3 = (1, 1, 1) \quad V_4 = (0, 1, 1) \tag{3.14}$$

The four zig-zag paths in Fig. 3 are the four possible combinations of two bifundamentals, $a_i b_j$. It follows that each bifundamental field corresponds to the intersection of two consecutive primitive normal vectors of the toric diagram. Furthermore in this case each external PM corresponds to one of the bifundamental fields. Again the charge and the flux assigned to each bifundamental field correspond to the charge and the flux assigned to each external PM

$$\begin{array}{llll} R_{a_1} = \Delta_{\pi_1}, & R_{b_1} = \Delta_{\pi_2}, & R_{a_2} = \Delta_{\pi_3}, & R_{b_2} = \Delta_{\pi_4} \\ n_{a_1} = n_{\pi_1} & n_{b_1} = n_{\pi_2} & n_{a_2} = n_{\pi_3} & n_{b_2} = n_{\pi_4} \end{array} \tag{3.15}$$

By substituting parameterization (3.15) in (3.13) we can check directly that the central charge c_r coincides with the one obtained from (3.5), under the conditions

$$\sum_{I=1}^{4} \Delta_{\pi_I} = 2, \qquad \sum_{I=1}^{4} n_{\pi_I} = 2\kappa \tag{3.16}$$

dP_2

We now consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_2 (see [20] for a discussion of the universal twist of dP_k theories). There are two Seiberg dual realizations of such a theory. Here we focus on the case with the minimal number of fields. This phase is usually referred to as the first phase and denoted as $dP_2^{(I)}$. It is a quiver gauge theory (see Fig. 4) with five $SU(N)$ gauge groups and superpotential

$$\begin{aligned} W = {} & X_{13}X_{34}X_{41} - Y_{12}X_{24}X_{41} + X_{12}X_{24}X_{45}Y_{51} - X_{13}X_{35}Y_{51} \\ & + Y_{12}X_{23}X_{35}X_{51} - X_{12}X_{23}X_{34}X_{45}X_{51}. \end{aligned} \tag{3.17}$$

The model has five non-anomalous abelian global symmetries. There are a $U(1)_R$ symmetry and two $U(1)$ flavor symmetries corresponding to the $U(1)^3$ isometry of the SE geometry. There are also five baryonic currents: Two of them are non-anomalous, two are anomalous and one is redundant.

We perform the c_r calculation from the geometry and we show the validity of formula (3.5) by matching the geometric result with the one obtained from the field theory analysis.

The dimer, the zig-zag paths and the toric diagram are shown in Fig. 5. The toric diagram is identified by the lattice points

$$V_1 = (1, 1, 1) \quad V_2 = (0, 1, 1) \quad V_3 = (-1, 0, 1) \quad V_4 = (-1, -1, 1) \quad V_5 = (0, -1, 1) \tag{3.18}$$

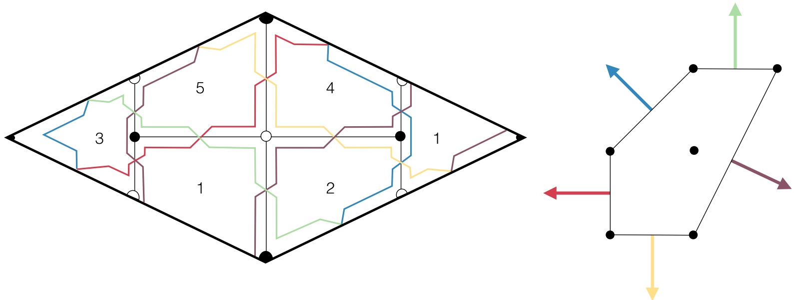


Fig. 5. Dimer, zig-zag paths and toric diagram of dP_2.

The R-charges and the fluxes of the fields can be parameterized in terms of the Δ_{π_I} charges and the n_{π_I} fluxes as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{13}$	$\Delta_{\pi_4} + \Delta_{\pi_5}$	$n_{\pi_4} + n_{\pi_5}$
$\phi_2 = X_{24}$	Δ_{π_5}	n_{π_5}
$\phi_3 = X_{51}$	Δ_{π_5}	n_{π_5}
$\phi_4 = X_{23}$	Δ_{π_2}	n_{π_2}
$\phi_5 = X_{41}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_6 = Y_{51}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_7 = Y_{12}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$
$\phi_8 = X_{45}$	Δ_{π_4}	n_{π_4}
$\phi_9 = X_{12}$	Δ_{π_1}	n_{π_1}
$\phi_{10} = X_{35}$	Δ_{π_1}	n_{π_1}
$\phi_{11} = X_{34}$	Δ_{π_3}	n_{π_3}

(3.19)

subject to the constraints $\sum_{I=1}^{5} \Delta_{\pi_I} = 2$ and $\sum_{I=1}^{5} n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$. In this case there are 5 gauge groups and the central charge is obtained from the formula

$$c_r = 3N^2 \eta_\Sigma \left(5\kappa + \sum_{i=1}^{11} (n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2 \right) \tag{3.20}$$

By substituting parameterization (3.19) in (3.20) we can see show that (3.20) is equivalent to (3.5) once the constraints (2.8) and (2.23) are imposed.

dP_3

Here we consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_3. There are four Seiberg dual realizations of such a theory, and we focus on the case with the minimal number of fields, usually called the first phase and denoted as $dP_3^{(I)}$. The quiver is represented in Fig. 6, and it has six gauge groups. The superpotential is

$$\begin{aligned} W = {} & X_{12}X_{24}X_{45}X_{51} - X_{24}X_{46}X_{62} + X_{23}X_{35}X_{56}X_{62} \\ & - X_{35}X_{51}X_{13} + X_{34}X_{46}X_{61}X_{13} - X_{12}X_{23}X_{34}X_{45}X_{56}X_{61}. \end{aligned} \tag{3.21}$$

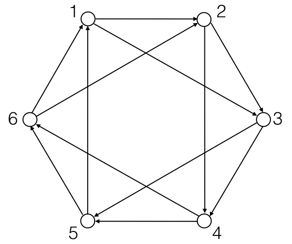


Fig. 6. Quiver of the $dP_3^{(I)}$ model.

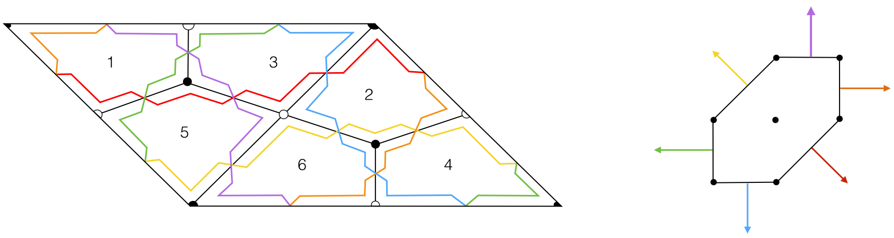


Fig. 7. Dimer, zig-zag paths and toric diagram of dP_3.

The model possesses six non-anomalous abelian global symmetries. There are a $U(1)_R$ symmetry and two $U(1)$ flavor symmetries corresponding to the $U(1)^3$ isometry of the SE geometry. There are also six baryonic currents: Three are non-anomalous, two are anomalous and one is redundant. The dimer, the zig-zag paths and the toric diagram are shown in Fig. 7. Again we can perform the calculation from the geometry, showing the validity of formula (3.5). The toric diagram is identified by the lattice points

$$
\begin{aligned}
&V_1 = (1, 1, 1) \qquad V_2 = (0, 1, 1) \qquad V_3 = (-1, 0, 1) \\
&V_4 = (-1, -1, 1) \quad V_5 = (0, -1, 1) \quad V_6 = (1, 0, 1)
\end{aligned}
\tag{3.22}
$$

The R-charges and the fluxes of the fields can be parameterized in terms of the Δ_{π_I} charges and of the n_{π_I} fluxes as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	Δ_{π_6}	n_{π_6}
$\phi_2 = X_{13}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_3 = X_{23}$	Δ_{π_5}	n_{π_5}
$\phi_4 = X_{24}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_5 = X_{34}$	Δ_{π_4}	n_{π_4}
$\phi_6 = X_{35}$	$\Delta_{\pi_1} + \Delta_{\pi_6}$	$n_{\pi_1} + n_{\pi_6}$
$\phi_7 = X_{45}$	Δ_{π_3}	n_{π_3}
$\phi_8 = X_{46}$	$\Delta_{\pi_5} + \Delta_{\pi_6}$	$n_{\pi_5} + n_{\pi_6}$
$\phi_9 = X_{56}$	Δ_{π_2}	n_{π_2}
$\phi_{10} = X_{51}$	$\Delta_{\pi_4} + \Delta_{\pi_5}$	$n_{\pi_4} + n_{\pi_5}$
$\phi_{11} = X_{61}$	Δ_{π_1}	n_{π_1}
$\phi_{12} = X_{62}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$

(3.23)

with the constraints $\sum_{I=1}^{6} \Delta_{\pi_I} = 2$ and $\sum_{I=1}^{6} n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a \in W} R_{\phi_a} = 2$ and $\sum_{a \in W} n_{\phi_a} = 2\kappa$. The central charge is obtained from the formula

$$c_r = 3N^2 \eta_\Sigma \left(6\kappa + \sum_{i=1}^{12} (n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2 \right) \tag{3.24}$$

By substituting parameterization (3.23) in (3.24) we can easily see that (3.24) is equivalent to (3.5) provided the constraints (2.8) and (2.23) are imposed.

Y^{pq} theories

We can prove the validity of (3.5) also for infinite families of quiver gauge theories. The first family that we consider is $X_5 = Y^{pq}$. These models has been derived in [23]. They are quiver gauge theories with $2p$ gauge groups and bifundamental matter. For generic values of p and q the models have a $SU(2) \times U(1)$ flavor symmetry and one non-anomalous baryonic $U(1)$ symmetry. At the 2d fixed point this baryonic symmetry generically mixes with the R-current.

The general prescription to obtain the exact 2d central charge after twisted compactification has been given in [20] and detailed explicitly there for some cases of particular interest. Knowing the field content of these theories as summarized in Table (3.27), at large N we can use the general formula (2.22) to write

$$c_r = 3N^2 \eta_\Sigma \left(2p\kappa + \sum_{i=1}^{6} d_i (n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2 \right) \tag{3.25}$$

We now show how to reproduce this expression from our geometric formulation (3.5).

For generic values of p and q the toric diagram has four external corners. There are also internal lattice points, associated to the anomalous baryonic symmetries, that do not play any role in our analysis. The corners of the toric diagram are associated to the vectors

$$V_1 = (0, 0, 1), \quad V_2 = (1, 0, 1), \quad V_3 = (0, p, 1) \quad V_4 = (-1, p - q, 1) \tag{3.26}$$

The parameterization of the R-charges and fluxes for the various fields in terms of the toric data can be read from the following table

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = Y$	$p+q$	Δ_{π_1}	n_{π_1}
$\phi_2 = U_1$	p	Δ_{π_2}	n_{π_2}
$\phi_3 = Z$	$p-q$	Δ_{π_3}	n_{π_3}
$\phi_4 = U_2$	p	Δ_{π_4}	n_{π_4}
$\phi_5 = V_1$	q	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_6 = V_2$	q	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$

(3.27)

The charges are subject to constraints (2.8). This parameterization satisfies the constraints $\sum_{a \in W} R_{\phi_a} = 2$ and $\sum_{a \in W} n_{\phi_a} = 2\kappa$ at each node of the dimer.

It is now easy to check that substituting these expressions for the R-charges and the fluxes in (3.25) and taking into account constraints (2.8) and (2.23) we reproduce exactly what we would obtain from (3.5).

L^{pqr} theories

We now consider a second infinite family, corresponding to $X_5 = L^{pqr}$, for $p \neq r$ (the degenerate case $p = r$ will be treated in section 4). These models have been derived in [45,54,55]. They can be described in terms of a necklace quiver, i.e. a set of $p+q$ $SU(N)$ gauge groups such that each node is connected to its nearest neighbors by a bifundamental and an anti-bifundamental fields. In general there may be also additional adjoint chiral multiplets, depending on the value of p and q and on the Seiberg dual phase that we are considering.

The central charge at large N can be easily obtained from (2.22) taking into account the field content of these theories in their the minimal phase, as summarized in table (3.30)

$$c_r = 3N^2\eta_\Sigma\left((p+q)\kappa + \sum_{i=1}^{6} d_i(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.28}$$

To check the equivalence with the geometric prescrition (3.5) we first assign the external corners of the toric diagrams to the following vectors

$$V_1 = (0,0,1), \quad V_2 = (1,0,1), \quad V_3 = (P,s,1) \quad V_4 = (-k,q,1) \tag{3.29}$$

where $r - Ps - kq = 0$ and $p + q = r + s$ and $p \leq r \leq q \leq s$. R-charges and fluxes parametrized in terms of the PM's are

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = Y$	q	Δ_{π_1}	n_{π_1}
$\phi_2 = W_2$	s	Δ_{π_2}	n_{π_2}
$\phi_3 = Z$	p	Δ_{π_3}	n_{π_3}
$\phi_4 = X_1$	r	Δ_{π_4}	n_{π_4}
$\phi_5 = W_1$	$q-s$	$\Delta_{\pi_2}+\Delta_{\pi_3}$	$n_{\pi_2}+n_{\pi_3}$
$\phi_6 = X_1$	$q-r$	$\Delta_{\pi_3}+\Delta_{\pi_4}$	$n_{\pi_3}+n_{\pi_4}$

(3.30)

with the constraints $\sum_{I=1}^4 \Delta_{\pi_I} = 2$ and $\sum_{I=1}^4 n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

Substituting this parameterization in (3.28) we directly obtain an expression equivalent to (3.5), once constraints (2.8) and (2.23) are taken into account.

X^{pq} theories

Finally we consider the infinite family of models corresponding to $X_5 = X^{pq}$. They have been constructed in [52]. In this case there are $2p+1$ gauge groups and taking into account the spectrum of fields and their multiplicities as given in table (3.32), the 2d central charge as read from (2.22) is

$$c_r = 3N^2\eta_\Sigma\left((2p+1)\kappa + \sum_{i=1}^{10} d_i(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.31}$$

To check it against the geometric calculation (2.22), we first label the external corners of the toric diagrams as (we take $p > q$)

$$V_1 = (1,p,1), \quad V_2 = (0,p-q+1,1), \quad V_3 = (0,p-q,1)$$
$$V_4 = (1,0,1) \quad V_5 = (2,0,1)$$

The R-charges and fluxes parametrization in terms of the PM's is given by

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
ϕ_1	$p+q-1$	Δ_{π_1}	n_{π_1}
ϕ_2	1	Δ_{π_2}	n_{π_2}
ϕ_3	1	Δ_{π_3}	n_{π_3}
ϕ_4	$p-q$	Δ_{π_4}	n_{π_4}
ϕ_5	p	Δ_{π_5}	n_{π_5}
ϕ_6	$p-1$	$\Delta_{\pi_2}+\Delta_{\pi_3}$	$n_{\pi_2}+n_{\pi_3}$
ϕ_7	1	$\Delta_{\pi_3}+\Delta_{\pi_4}$	$n_{\pi_3}+n_{\pi_4}$
ϕ_8	$q-1$	$\Delta_{\pi_2}+\Delta_{\pi_3}+\Delta_{\pi_4}$	$n_{\pi_2}+n_{\pi_3}+n_{\pi_4}$
ϕ_9	1	$\Delta_{\pi_1}+\Delta_{\pi_2}$	$n_{\pi_1}+n_{\pi_2}$
ϕ_{10}	q	$\Delta_{\pi_4}+\Delta_{\pi_5}$	$n_{\pi_4}+n_{\pi_5}$

(3.32)

with the constraints $\sum_{I=1}^{5}\Delta_{\pi_I}=2$ and $\sum_{I=1}^{5}n_{\pi_I}=2\kappa$. Once again, this parameterization satisfies the constraints $\sum_{a\in W}R_{\phi_a}=2$ and $\sum_{a\in W}n_{\phi_a}=2\kappa$.

Using this parameterization it is easy to check that result (3.31) is equivalent to (3.5), once constraints (2.8) and (2.23) are imposed.

4. Singular horizons and lattice points lying on the perimeter

In this section we discuss the case of toric diagrams with some external lattice points that are not corners but lie along the perimeter. These diagrams are associated to theories with non-smooth horizons, usually arising from the action of an orbifold.

In this case, as discussed in [5], the geometric procedure to extract the central charge a from the toric diagram needs some modification. The reason is that the lattice points lying on the perimeter are associated to a multiple number of PM's. Therefore, this requires a change in the prescription for assigning R-charges to the fields in terms of the charges of the PM's.

The prescription that we propose follows the one described in [33] and it works as follows. First divide the PM's in two sets, the ones associated to corners of the toric diagram and the degenerate ones lying on the perimeter, namely π^c and π^p respectively. Then we associate a R-charge $\Delta_{\pi_I^c}$ to the PM's at the corners, as done before. For the PM's on the perimeter, observing that at each point on the perimeter only one of the degenerate PM's enters the definition of the zig-zag paths, we assign a non-zero charge $\Delta_{\pi_I^p}$ to this PM and set the charge of all the other PM's associated to the same I-th lattice point to zero. With this modification of charge assignments we can then parameterize the R-charges R_i and the fluxes n_i unambiguously as described in section 3.

We have checked in a large set of examples that by applying this prescription the 2d central charge computed from the field theory analysis, eq. (2.22), matches with the one computed using formula (3.5). In the following we report the explicit check for a couple of examples in the L^{pqp} class.

4.1. L^{222}

For this particular representative of the L^{pqr} family the quiver diagram, the dimer with the zig-zag paths and the toric diagram are depicted in Fig. 8. The superpotential of this model is

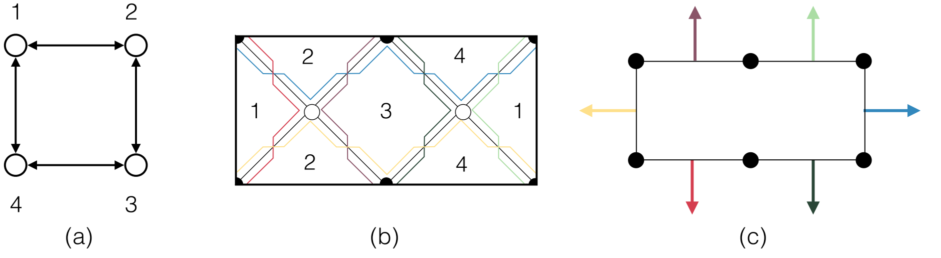


Fig. 8. Dimer, zig zag paths and toric diagram of L^{222}.

$$W = X_{12}X_{23}X_{32}X_{21} - X_{23}X_{34}X_{43}X_{32} + X_{34}X_{41}X_{14}X_{43} - X_{41}X_{12}X_{21}X_{14} \tag{4.1}$$

The central charge can be obtained from formula (2.22) once we take into account the specific field content of the theory that can be read from the quiver diagram or in table (4.5). We obtain

$$c_r = 3N^2\eta_\Sigma \left(4\kappa + \sum_{i=1}^{8}(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{4.2}$$

In order to match this expression with (3.5) we first observe that the PM's are related to the lattice points as follows

PM	Lattice point	PM	Lattice point
$\pi_1 = \{X_{12}, X_{34}\}$	$V_1 = (0, 0, 1)$	$\pi_2 = \{X_{21}, X_{34}\}$	$V_2 = (1, 0, 1)$
$\pi_3 = \{X_{12}, X_{43}\}$	$V_2 = (1, 0, 1)$	$\pi_4 = \{X_{21}, X_{43}\}$	$V_3 = (2, 0, 1)$
$\pi_5 = \{X_{32}, X_{14}\}$	$V_4 = (2, 1, 1)$	$\pi_6 = \{X_{32}, X_{41}\}$	$V_5 = (1, 1, 1)$
$\pi_7 = \{X_{23}, X_{14}\}$	$V_5 = (1, 1, 1)$	$\pi_8 = \{X_{23}, X_{41}\}$	$V_6 = (0, 1, 1)$

(4.3)

The two points on the perimeter, identified as V_2 and V_5, are degenerate since they correspond to two different PM's. According to our prescription in sub-section 3.1, we set $\Delta_{\pi_3} = \Delta_{\pi_7} = 0$ and $n_{\pi_3} = n_{\pi_7} = 0$. The other non-vanishing charges and fluxes are constrained by the relations

$$\begin{array}{ccccccccccccc} \Delta_{\pi_1} & + & \Delta_{\pi_2} & + & \Delta_{\pi_4} & + & \Delta_{\pi_5} & + & \Delta_{\pi_6} & + & \Delta_{\pi_8} & = & 2 \\ n_{\pi_1} & + & n_{\pi_2} & + & n_{\pi_4} & + & n_{\pi_5} & + & n_{\pi_6} & + & n_{\pi_8} & = & 2\kappa \end{array} \tag{4.4}$$

From here we can read the charges and the fluxes of every single field

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	Δ_{π_1}	n_{π_1}
$\phi_2 = X_{21}$	$\Delta_{\pi_4} + \Delta_{\pi_2}$	$n_{\pi_4} + n_{\pi_2}$
$\phi_3 = X_{23}$	Δ_{π_8}	n_{π_8}
$\phi_4 = X_{32}$	$\Delta_{\pi_5} + \Delta_{\pi_6}$	$n_{\pi_5} + n_{\pi_6}$
$\phi_5 = X_{34}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_6 = X_{43}$	Δ_{π_4}	n_{π_4}
$\phi_7 = X_{41}$	$\Delta_{\pi_6} + \Delta_{\pi_8}$	$n_{\pi_6} + n_{\pi_8}$
$\phi_8 = X_{14}$	Δ_{π_5}	n_{π_5}

(4.5)

This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

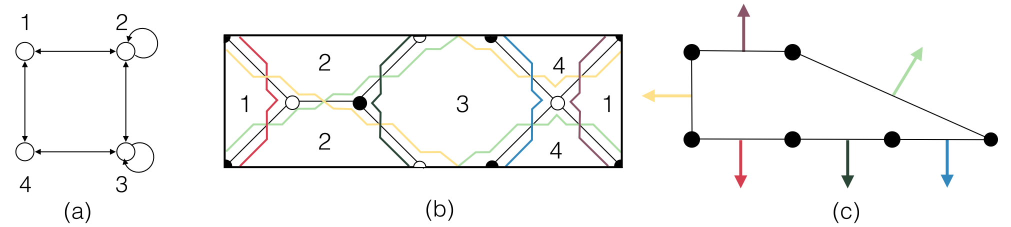


Fig. 9. Dimer, zig-zag paths and toric diagram of L[131].

By substituting this parametrization in (4.2) we can easily prove that it is equivalent to (3.5) once constraints (4.4) are imposed.

4.2. L^{131}

As a second example, we consider the L^{131} model associated to the quiver, dimer and toric diagram drawn in Fig. 9. In this case the superpotential reads

$$\begin{aligned} W = & X_{12}X_{21}X_{14}X_{41} - X_{12}X_{22}X_{21} + X_{32}X_{22}X_{23} \\ & - X_{23}X_{33}X_{32} + X_{43}X_{33}X_{34} - X_{14}X_{43}X_{34}X_{41} \end{aligned} \tag{4.6}$$

Given the particular field content, the central charge computed from (2.22) reads

$$c_r = 3N^2\eta_\Sigma \left(4\kappa + \sum_{i=1}^{10}(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{4.7}$$

In this case the PM's are related to the lattice points as follows

PM	Lattice point	PM	Lattice point
$\pi_1 = \{X_{12}, X_{23}, X_{34}\}$	$V_1 = (0, 0, 1)$	$\pi_2 = \{X_{21}, X_{23}, X_{34}\}$	$V_2 = (1, 0, 1)$
$\pi_3 = \{X_{12}, X_{32}, X_{34}\}$	$V_2 = (1, 0, 1)$	$\pi_4 = \{X_{12}, X_{23}, X_{43}\}$	$V_2 = (1, 0, 1)$
$\pi_5 = \{X_{21}, X_{32}, X_{34}\}$	$V_3 = (2, 0, 1)$	$\pi_6 = \{X_{21}, X_{23}, X_{43}\}$	$V_3 = (2, 0, 1)$
$\pi_7 = \{X_{12}, X_{32}, X_{43}\}$	$V_3 = (2, 0, 1)$	$\pi_8 = \{X_{21}, X_{32}, X_{43}\}$	$V_4 = (3, 0, 1)$
$\pi_9 = \{X_{14}, X_{22}, X_{33}\}$	$V_5 = (1, 1, 1)$	$\pi_{10} = \{X_{41}, X_{22}, X_{33}\}$	$V_6 = (0, 1, 1)$

(4.8)

There are still two perimeter points, this time with degeneracy three. We set $\Delta_{\pi_2} = \Delta_{\pi_3} = \Delta_{\pi_5} = \Delta_{\pi_6} = 0$ and correspondingly $n_{\pi_2} = n_{\pi_3} = n_{\pi_5} = n_{\pi_6} = 0$. The remaining charges and fluxes satisfy

$$\begin{array}{ccccccccccccc} \Delta_{\pi_1} & + & \Delta_{\pi_4} & + & \Delta_{\pi_7} & + & \Delta_{\pi_8} & + & \Delta_{\pi_9} & + & \Delta_{\pi_{10}} & = & 2 \\ n_{\pi_1} & + & n_{\pi_4} & + & n_{\pi_7} & + & n_{\pi_6} & + & n_{\pi_9} & + & n_{\pi_{10}} & = & 2\kappa \end{array} \tag{4.9}$$

The R-charges R_{ϕ_i} and the fluxes n_{ϕ_i} of the fields can be expressed in terms of the charges Δ_{π_I} and the fluxes n_{π_I} of the PM's as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	$\Delta_{\pi_1} + \Delta_{\pi_4} + \Delta_{\pi_7}$	$n_{\pi_1} + n_{\pi_4} + n_{\pi_7}$
$\phi_2 = X_{21}$	Δ_{π_8}	n_{π_8}
$\phi_3 = X_{22}$	$\Delta_{\pi_9} + \Delta_{\pi_{10}}$	$n_{\pi_9} + n_{\pi_{10}}$
$\phi_4 = X_{23}$	$\Delta_{\pi_1} + \Delta_{\pi_4}$	$n_{\pi_1} + n_{\pi_4}$
$\phi_5 = X_{32}$	$\Delta_{\pi_7} + \Delta_{\pi_8}$	$n_{\pi_7} + n_{\pi_8}$
$\phi_6 = X_{33}$	$\Delta_{\pi_9} + \Delta_{\pi_{10}}$	$n_{\pi_9} + n_{\pi_{10}}$
$\phi_7 = X_{34}$	Δ_{π_1}	n_{π_1}
$\phi_8 = X_{43}$	$\Delta_{\pi_4} + \Delta_{\pi_7} + \Delta_{\pi_8}$	$n_{\pi_4} + n_{\pi_7} + n_{\pi_8}$
$\phi_9 = X_{41}$	$\Delta_{\pi_{10}}$	$n_{\pi_{10}}$
$\phi_{10} = X_{14}$	Δ_{π_9}	n_{π_9}

(4.10)

This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

It is now easy to substitute this parameterization in (4.7) and check that the resulting expression is equivalent to (3.5) once we take into account constraints (4.9).

5. Mixing of the baryonic symmetries

In this section, by studying the twisted compactification of some of the 4d $\mathcal{N} = 1$ toric quiver gauge theories discussed above, we provide further evidence that both flavor and baryonic symmetries mix with the R-current at the 2d fixed point. We compute the central charge with the formalism reviewed in section 2.2 showing its positivity for many choices of the curvature and the fluxes.

dP$_2$

We begin by identifying the global currents of the dP$_2$ model. There are a UV R-current R_0, two flavor currents $F_{1,2}$ and two non-anomalous baryonic currents $B_{1,2}$. Having the model five gauge groups, to begin with we have five classically conserved baryonic currents, associated to the decoupling of the gauge abelian factors $U(1)_i \subset U(N)_i$. As usual one of such currents is redundant. Among the other global baryonic $U(1)$'s some of the combinations can be anomalous at quantum level. After the identification of the two non-anomalous baryonic currents the charges of the fields respect to all the global currents are

	Y_{51}	X_{51}	X_{23}	X_{35}	X_{41}	X_{34}	X_{13}	X_{24}	X_{45}	X_{12}	Y_{12}
R_0	2	0	2	0	2	0	0	0	0	0	0
F_1	−2	1	−3	1	−2	1	1	1	0	1	1
F_2	1	1	−1	1	0	2	−2	1	−3	1	−1
B_1	−1	−1	−1	1	0	0	0	−1	1	1	1
B_2	1	1	−1	0	−1	2	−1	1	−2	0	0

(5.1)

Any linear combination

$$R_{\text{trial}} = R_0 + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2} + \eta_1 T_{B_1} + \eta_2 T_{B_2} \tag{5.2}$$

is still an R-current. Such an ambiguity is fixed by maximizing the central charge with respect to the mixing parameters ϵ_i and η_i [4]

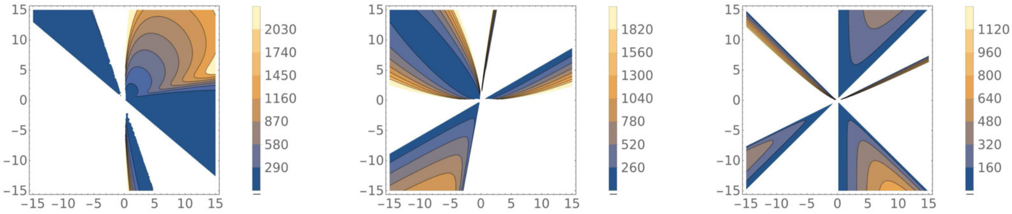


Fig. 10. Central charge of dP_2 on $\Sigma = \mathbb{T}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

$$\frac{\partial a}{\partial \epsilon_i} = \frac{3}{32}[9\,\mathrm{Tr}(R_{\mathrm{trial}}^2 F_i) - \mathrm{Tr}(F_i)] = 0 \tag{5.3}$$

$$\frac{\partial a}{\partial \eta_i} = \frac{3}{32}[9\,\mathrm{Tr}(R_{\mathrm{trial}}^2 B_i) - \mathrm{Tr}(B_i)] = 0. \tag{5.4}$$

By using the relations $\mathrm{Tr}(B_i B_j B_k) = 0 = \mathrm{Tr}(B_i)$ equations (5.4) reduce to a linear system in the η_i variables. Substituting the solution back into (5.3) we are left with two free mixing parameters. Therefore, one can always linearly combine the global symmetries in such a way that at the fixed point $\eta_1 = \eta_2 = 0$. This signals the fact that the baryonic symmetries do not mix with the 4d exact R-current.

Solving the rest of equations we obtain

$$\epsilon_1 = \frac{1}{8}\left(\sqrt{33} - 1\right), \quad \epsilon_2 = \frac{1}{16}\left(3\sqrt{33} - 19\right) \tag{5.5}$$

We can proceed by twisting the theory on Σ. The partial topological twist is performed along the generator

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + b_3 T_{B_1} + b_4 T_{B_2} \tag{5.6}$$

The central charge c_r of the 2d theory can be obtained from (2.18). The final formulae are quite involved and we report some non-trivial cases in appendix A. In this case we observe that the mixing parameters are non-vanishing for generic choices of the constant curvature and of the fluxes $b_{\mathbf{I}}$. This signals the fact that the baryonic symmetries mix with the 2d exact R-current.

We conclude by showing in Fig. 10, 11 and 12 the central charge for different values of the discrete fluxes for dP_2 compactified on $\Sigma = \mathbb{T}^2$, $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{H}^2$, respectively. The scale of colors represents the value of the central charge in units of N^2.

dP_3

We now consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_3. The global currents of the model are a UV R-current R_0, two flavor currents $F_{1,2}$ and three non-anomalous baryonic currents $B_{1,2,3}$.

We can identify the baryonic currents as follows. The model has six gauge groups and classically there are six conserved baryonic currents. Two of them are anomalous and one is redundant. One is left with three non-anomalous baryonic symmetries. We can choose the charges of the fields with respect of the global symmetries as

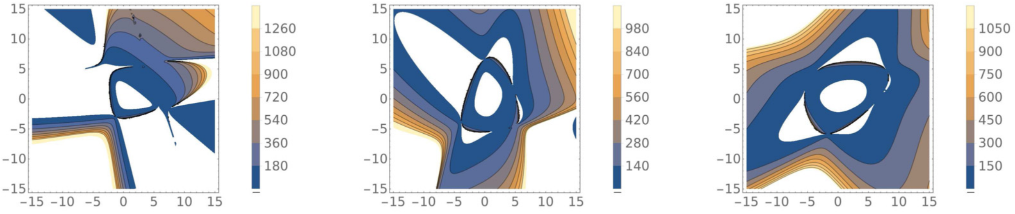


Fig. 11. Central charge of dP_2 on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

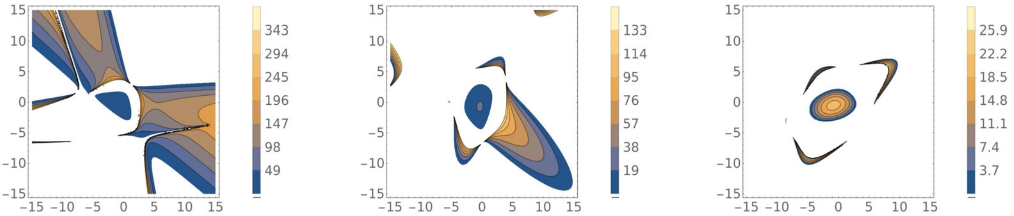


Fig. 12. Central charge of dP_2 on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

	X_{12}	X_{13}	X_{23}	X_{24}	X_{34}	X_{35}	X_{45}	X_{46}	X_{56}	X_{51}	X_{61}	X_{62}
R_0	2	0	0	0	0	2	0	2	0	0	0	0
F_1	−1	1	0	−1	1	−2	1	−1	0	1	−1	2
F_2	3	0	−1	−4	1	0	1	2	−1	0	−3	2
B_1	2	1	−1	−1	0	0	0	1	1	−1	−2	0
B_2	1	0	−1	−1	0	1	1	0	−1	−1	0	1
B_3	−1	−1	0	1	1	0	−1	−1	0	1	1	0

Any linear combination

$$R_{\text{trial}} = R_0 + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2} + \eta_1 T_{B_1} + \eta_2 T_{B_2} + \eta_3 T_{B_3} \tag{5.7}$$

is still an R-current. The mixing coefficients in (5.7) are fixed by maximizing the central charge, and in this case we find

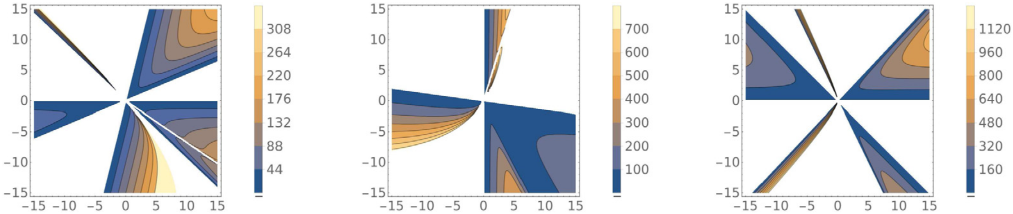
$$\epsilon_1 = \frac{2}{3}, \quad \epsilon_2 = -\frac{1}{3}, \quad \eta_i = 0 \tag{5.8}$$

Again we chose a parameterization such that at the superconformal fixed point the contribution of the baryonic symmetries vanishes.

The partial topological twist on Σ is performed along the generator

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + b_3 T_{B_1} + b_4 T_{B_2} + b_5 T_{B_3} \tag{5.9}$$

The central charge c_r is obtained from (2.18). The final expressions are too complicated and we do not learn much in writing them explicitly. The main point is that, as in the case of dP_2, the mixing parameters are non-vanishing (see appendix A for some examples) for generic choices of



$b_1 = x,\, b_2 = y$
$b_3 = b_4 = b_5 = 0$

$b_3 = x,\, b_4 = y$
$b_1 = b_2 = b_5 = 0$

$b_2 = x,\, b_3 = y$
$b_1 = b_4 = b_5 = 0$

Fig. 13. Central charge of dP_3 on $\Sigma = \mathbb{T}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. The variable x is plotted on the horizontal axis while y on the vertical one.

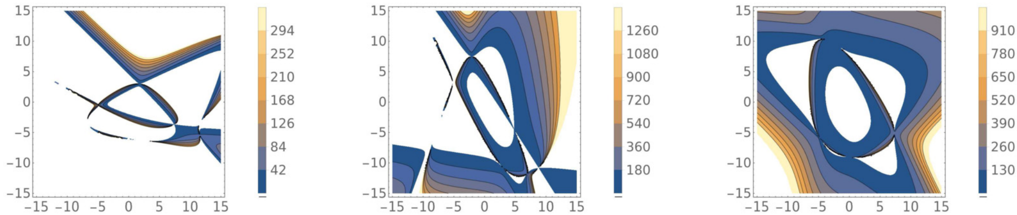


Fig. 14. Central charge of dP_3 on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = b_5 = 0$.

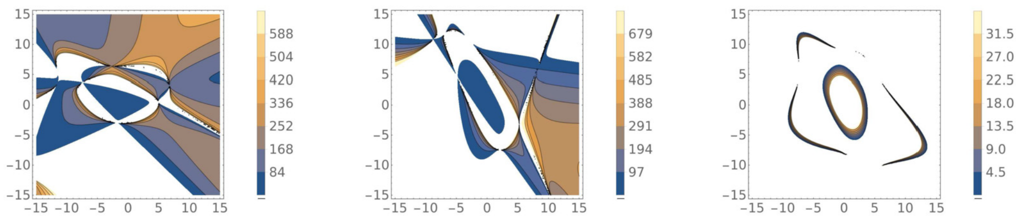


Fig. 15. Central charge of dP_3 on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = b_5 = 0$.

the curvature and the $b_\mathbf{I}$ fluxes. This signals the fact that the baryonic symmetries mix with the 2d exact R-current.

We conclude by showing in Fig. 13, 14 and 15 the central charge for different values of the discrete fluxes for dP_2 compactified on $\Sigma = \mathbb{T}^2$, $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{H}^2$, respectively.

L^{pqp} *theories*

As a last example we consider models with a higher number of baryonic symmetries, i.e. L^{pqp} models [45,54,55]. In order to have a comprehensive discussion we pick up a particular (Seiberg dual) phase, that can be easily visualized by the description of the system in terms of D4 and NS branes in type IIA string theory (the other phases are obtained by exchanging the NS branes). We consider a stack of N D4 branes extended along x_{0123} and wrapping the compact direction x_6.

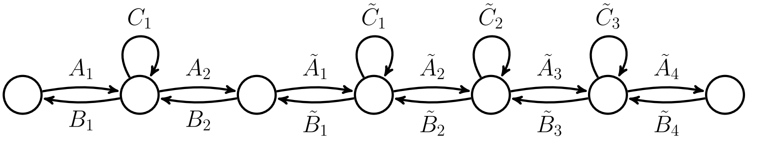


Fig. 16. Quiver of the L^{242} model. The first and the last node are identified.

Then we consider two sets of p NS and q NS' branes. The NS branes are extended along x_{012345} and the NS' along x_{012389}. We order the NS branes and then the NS' branes clockwise along x_6.

Each gauge group is associated to a segment of N D4 branes on x_6, suspended between two consecutive NS branes. The resulting field theory is a $U(N)$ necklace quiver gauge theory with different types of nodes. By counting clockwise on x_6 we have

- a set of $p-1$ nodes with an adjoint of type C;
- a node without any adjoint;
- a set of $q-1$ nodes with an adjoint of type $\tilde{C}$;
- a node without any adjoint.

The $2p$ bifundamental matter fields crossing a NS brane are of type A or B depending on their orientation and the second set of $2q$ bifundamental fields, crossing a NS' brane, are of type $\tilde{A}$ and $\tilde{B}$. We can visualize the situation in the quiver of Fig. 16 for the case or $p=2$ and $q=4$.

These theories are characterized by one R-current, two flavor currents and $p+q$ baryonic currents [45,54,55]. One baryonic current is redundant, being the quiver necklace. The vector-like nature of the field content ensures that the other $p+q-1$ currents are all conserved at quantum level. The charge assignment of flavor and R-currents is summarized in the following table[3]

field	mult	F_1	F_2	R_0
A	p	0	1	0
B	p	-1	0	0
$\tilde{A}$	q	1	0	1
$\tilde{B}$	q	0	-1	1
C	$p-1$	1	-1	2
$\tilde{C}$	$q-1$	-1	1	0

(5.10)

The baryonic currents are associated to the $U(1)_i \subset U(N)_i$ gauge factors and the charges are read from the representation of each field under the gauge groups. Fundamental fields of $SU(N)_i$ have charge $+1$ and anti-fundamental fields of $SU(N)_i$ have charge -1 under the baryonic $U(1)_i$.

The 4d R-charge mixes with the global symmetries through the combination

$$R = R_0 + \epsilon_1 F_1 + \epsilon_2 T_{F_2} + \sum_{i=1}^{p+q-1} \eta_i T_{B_i} \tag{5.11}$$

with mixing parameters

$$\epsilon_1 = \frac{\sqrt{p^2 - pq + q^2} - 2p + q}{3(p-q)}, \quad \epsilon_2 = \frac{p}{\sqrt{p^2 - pq + q^2} + 2p - q}, \quad \eta_i = 0 \tag{5.12}$$

determined by the a-maximization.

[3] We refer to R_0 as a trial R-charge obtained after the maximization on the baryonic charges, as described after (5.4).

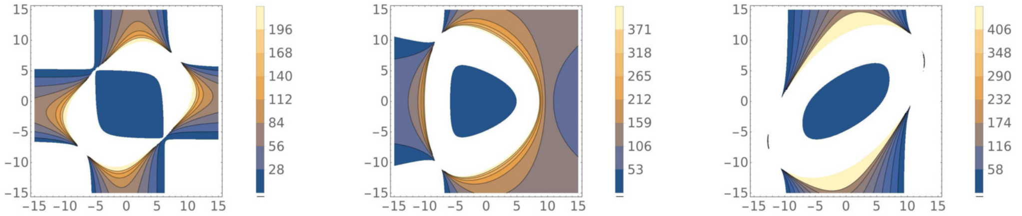


Fig. 17. Central charge of L^{222} on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

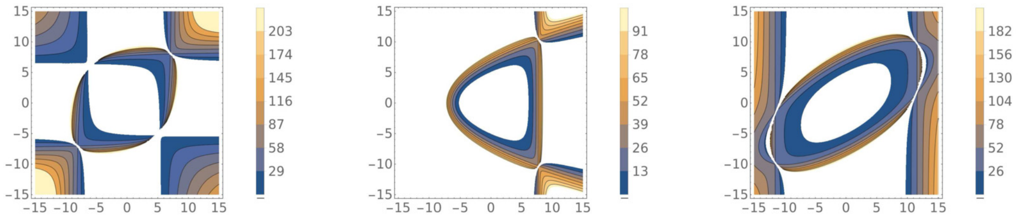


Fig. 18. Central charge of L^{222} on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

When the theory is partially topologically twisted on Σ along the generator

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + \sum_{i=1}^{p+q-1} b_{i+2} B_i \tag{5.13}$$

the central charge c_r is obtained from (2.18) and extremized respect to the ϵ_I parameters. The final formulas are too involved and we do not report them here. The mixing parameters are non-vanishing for generic choices of the curvature and of the fluxes $b_{\mathbf{I}}$. This signals the fact that the baryonic symmetries mix with the 2d exact R-current.

In the following we show some numerical result for the 2d central charge for the L^{222} and the L^{131} gauge theories. In both cases there are four gauge groups and three non-anomalous baryonic symmetries. In both case we observe that the baryonic symmetries mix for generic values of the with the R-current at the 2d fixed point. In Fig. 17 and 18 we represent the central charge for different values of the discrete fluxes for L^{222} compactified on $\Sigma = \mathbb{H}^2$ and $\Sigma = \mathbb{S}^2$, respectively.

In the case of the torus reduction ($\kappa = 0$) the formulae are simpler and we can provide the analytical expression for c_r extremized with respect to the mixing parameters, in terms of the b_i fluxes

$$c_r^{L^{222},T^2} = \frac{6(b_3^2+b_1b_3+b_2b_3-2b_4b_3+2b_4^2+b_5^2+2b_1b_2-(b_1+b_2+2b_4)b_5)(b_3^2+b_5^2+b_2(b_5-b_3)+b_1(2b_2-b_3+b_5))}{(b_1-b_2)(b_4^2-b_3b_4+b_1b_3+b_2b_3-(b_1+b_2+b_4)b_5)} \tag{5.14}$$

The parameters ϵ_i^* are generically non-vanishing for both the flavor and the baryonic global symmetries.

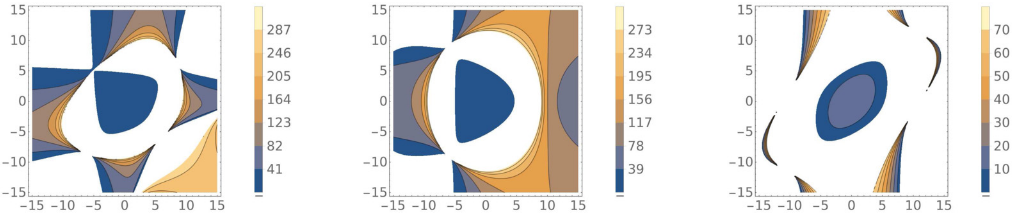


Fig. 19. Central charge of L^{131} on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

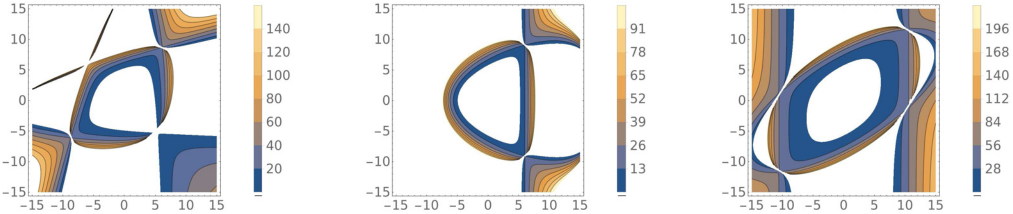


Fig. 20. Central charge of L^{131} on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

In Fig. 19 and 20 we represent the central charge for different values of the discrete fluxes for L^{131} compactified on $\Sigma = \mathbb{H}^2$ and $\Sigma = \mathbb{S}^2$ respectively.

In the case of the torus reduction ($\kappa = 0$) the formulae are simpler and we can provide the analytical expression for c_r extremized with respect to the mixing parameters, in terms of the b_i fluxes

$$c_r^{L^{131},T^2} = 6\left(\frac{(b_1^2-3b_2b_1+b_2^2-b_3^2+(b_1+b_2)b_3)^2}{(b_1-b_2)((b_1^2+(b_3-b_2)b_1+b_2^2+b_4^2+b_5^2+b_2b_3-b_3b_4-b_4b_5)} - \frac{b_1^2+4b_2b_1-2b_3b_1-b_2^2+2b_3^2-2b_2b_3}{b_1-b_2}\right) \tag{5.15}$$

Also in this case we observe that the parameters ϵ_i^* are generically non-vanishing for both the flavor and the baryonic global symmetries.

6. Further directions

In this paper we have studied c-extremization for 2d $\mathcal{N} = (0, 2)$ SCFTs arising from the twisted compactification on compact, constant curvature Riemann surfaces of infinite families of quiver gauge theories holographically dual to D3 branes probing the tip of CY_3 cones over X_5 basis admitting a $U(1)^3$ toric action. In such cases we have been able to develop a simple geometric formulation for the 2d trial central charge c_r, which in the large N limit turns out to be expressible in terms of the geometric data of the toric diagram of the 4d parent theory (see eq. (3.5)). Our result represents the field theory dual of the holographic formula found in [15,38,39].

This formulation borrows many ideas and constructions developed in the 4d parent theory, in which it has been demonstrated that the conformal anomaly a is proportional to the inverse of the X_5 volume. The geometric analogy with the 4d formulation that we have discussed may be helpful in understanding the possible relation between c_r and the volume of the seven manifold [18,19] in the conjectured $AdS_3 \times \mathcal{M}_7$ correspondence (or eight manifolds in M-theory). It should be interesting to investigate this direction further.

In our analysis we have shown that the 2d central charge, expressed in terms of the mixing parameters, can be reformulated in the language of the toric geometry underlining the moduli space of the 4d theory. Nevertheless we did not give a general discussion on the extremization of this function. This point certainly deserves a separate and deep analysis. Indeed, the existence of an extremum is not guaranteed, as discussed in [7]. The main obstructions are due to the absence of a normalizable vacuum of the 2d CFT and to the presence of accidental symmetries at the IR fixed point. The study of this problem would be simplified by the knowledge of the spectrum and the interactions of the 2d models. Progresses in such directions have been made in [11,12,56]. On the geometric side it would be interesting to see if some of the tools developed in 4d (e.g. the zonotope discussed in [35]) can be useful for the analysis of the extremization properties of the 2d central charge.

As a last comment we wish to mention that recently infinite families of 2d SCFTs have been obtained by exploiting the role of the toric geometry [57,58]. These theories, denoted as brane brick models, are expected to describe the worldvolume theory of stacks of D1 branes probing the tip of toric CY_4 cones in type IIB. It has been shown that in such cases the toric geometry can be used to obtain the elliptic genus [59]. It would be interesting to further explore the role of toric geometry in these 2d SCFTs and look for possible connections, if any, with our results.

Acknowledgements

We are grateful to Marcos Crichigno and Domenico Orlando for useful discussions and comments. The work of A.A. is supported by the Swiss National Science Foundation (snf) under grant number pp00p2_157571/1. This work has been supported in part by Italian Ministero dell'Istruzione, Università e Ricerca (MIUR) and Istituto Nazionale di Fisica Nucleare (INFN) through the "Gauge Theories, Strings, Supergravity" (GSS) research project. We thank the Galileo Galilei Institute for Theoretical Physics (GGI) for the hospitality and INFN for partial support during the completion of this work, within the program "New Developments in AdS_3/CFT_2 Holography".

Appendix A. Mixing parameters for dP_2 and dP_3

In this appendix we report the value of the mixing parameters for some choices of fluxes for the dP_2 and the dP_3 models studied in the body of the paper. The general results are pretty involved, so we restrict to some simple choices of fluxes $b_{\mathbf{I}}$.

In the dP_2 case we just show the $\kappa = 0$ case and one non-vanishing flux each time. By following the notations of section 5 we refer to ϵ as the mixing parameters of the flavor symmetries and to η as the mixing parameters of the baryonic symmetries. We have the following cases

- $b_1 \neq 0$

$$\epsilon_1 = \frac{7b_1^4 + 85b_1^3 - 1800b_1^2 - 4500b_1 + 60000}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}$$

$$
\begin{aligned}
\epsilon_2 &= \frac{11b_1^4 + 110b_1^3 - 225b_1^2 - 3250b_1 - 10000}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000} \\
\eta_1 &= -\frac{3b_1^4 - 20b_1^3 - 525b_1^2 + 2000b_1}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000} \\
\eta_2 &= -\frac{18b_1^4 - 10b_1^3 - 600b_1^2 - 500b_1}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}
\end{aligned} \tag{A.1}
$$

- $b_2 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= -\frac{8b_2^4 - 270b_2^3 + 2600b_2^2 - 1250b_2 - 45000}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000} \\
\epsilon_2 &= -\frac{8b_2^3 - 175b_2^2 + 525b_2 + 500}{12b_2^3 - 255b_2^2 + 300b_2 + 5000} \\
\eta_1 &= -\frac{4b_2^4 - 135b_2^3 + 850b_2^2 - 1000b_2}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000} \\
\eta_2 &= -\frac{8b_2^4 - 270b_2^3 + 2000b_2^2 + 250b_2}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000}
\end{aligned} \tag{A.2}
$$

- $b_3 \neq 0$

$$
\epsilon_1 = \frac{3}{5}, \quad \epsilon_2 = -\frac{1}{10}, \quad \eta_1 = -\frac{b_3}{10}, \quad \eta_2 = 0 \tag{A.3}
$$

- $b_4 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= \frac{95b_4^3 - 900b_4^2 + 45000}{3b_4^4 - 1950b_4^2 + 1500b_4 + 75000} \\
\epsilon_2 &= -\frac{4b_4^4 + 5b_4^3 - 1000b_4^2 + 1000b_4 + 5000}{2b_4^4 - 1300b_4^2 + 1000b_4 + 50000} \\
\eta_2 &= -\frac{12b_4^4 + 175b_4^3 + 750b_4^2 - 1500b_4}{6b_4^4 - 3900b_4^2 + 3000b_4 + 150000} \\
\eta_2 &= \frac{6b_4^4 + 80b_4^3 - 1800b_4^2 - 6000b_4}{3b_4^4 - 1950b_4^2 + 1500b_4 + 75000}
\end{aligned} \tag{A.4}
$$

It is interesting to observe that in each case all the mixing parameters are non-vanishing, showing the general fact that the baryonic symmetries have a non-trivial mix with the R-current in 2d.

In the dP_3 case we consider the case of generic κ ($\kappa^2 = 1$) and again we fix only one non-vanishing flux for each case. We have the following cases

- $b_1 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= \frac{16979b_1 - 6982b_1^2\kappa - 740b_1^3 + 171360\kappa}{52998b_1 - 4416b_1^2\kappa - 1104b_1^3 + 235116\kappa} \\
\epsilon_2 &= -\frac{8165b_1 - 844b_1^2\kappa - 188b_1^3 + 42840\kappa}{26499b_1 - 2208b_1^2\kappa - 552b_1^3 + 117558\kappa}
\end{aligned}
$$

$$\begin{aligned}
\eta_1 &= -\frac{105b_1 + 16b_1^2\kappa + 4b_1^3 + 630\kappa}{8833b_1 - 736b_1^2\kappa - 184b_1^3 + 39186\kappa} \\
\eta_2 &= \frac{920b_1 - 1948b_1^2\kappa - 464b_1^3 - 630\kappa}{26499b_1 - 2208b_1^2\kappa - 552b_1^3 + 117558\kappa} \\
\eta_3 &= \frac{605b_1 - 1996b_1^2\kappa - 476b_1^3 - 2520\kappa}{52998b_1 - 4416b_1^2\kappa - 1104b_1^3 + 235116\kappa}
\end{aligned} \tag{A.5}$$

- $b_2 \neq 0$

$$\begin{aligned}
\epsilon_1 &= -\frac{4\left(5640b_2 - 41298b_2^2\kappa + 840b_2^3 + 737b_2^4\kappa - 52b_2^5 + 514080\kappa\right)}{3\left(43536b_2 + 116452b_2^2\kappa - 1952b_2^3 - 2053b_2^4\kappa + 104b_2^5 - 940464\kappa\right)} \\
\epsilon_2 &= \frac{199452b_2 - 177054b_2^2\kappa + 1767b_2^3 + 4138b_2^4\kappa - 338b_2^5 + 1028160\kappa}{130608b_2 + 349356b_2^2\kappa - 5856b_2^3 - 6159b_2^4\kappa + 312b_2^5 - 2821392\kappa} \\
\eta_1 &= \frac{51420b_2 + 15446b_2^2\kappa - 8649b_2^3 - 1289b_2^4\kappa + 156b_2^5 + 15120\kappa}{43536b_2 + 116452b_2^2\kappa - 1952b_2^3 - 2053b_2^4\kappa + 104b_2^5 + 940464\kappa} \\
\eta_2 &= \frac{960b_2 + 66888b_2^2\kappa - 39906b_2^3 - 4054b_2^4\kappa + 728b_2^5 + 15120\kappa}{130608b_2 + 349356b_2^2\kappa - 5856b_2^3 - 6159b_2^4\kappa + 312b_2^5 - 2821392\kappa} \\
\eta_3 &= \frac{79500b_2 + 63198b_2^2\kappa - 30447b_2^3 - 4835b_2^4\kappa + 364b_2^5 + 30240\kappa}{3\left(43536b_2 + 116452b_2^2\kappa - 1952b_2^3 - 2053b_2^4\kappa + 104b_2^5 - 940464\kappa\right)}
\end{aligned} \tag{A.6}$$

- $b_3 \neq 0$

$$\begin{aligned}
\epsilon_1 &= -\frac{15660b_3\kappa - 19524b_3^2 + 1857b_3^3\kappa - 41b_3^4 + 2056320}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\epsilon_2 &= \frac{18180b_3\kappa - 21588b_3^2 + 3729b_3^3\kappa + 41b_3^4 + 1028160}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\eta_1 &= \frac{107484b_3\kappa - 3564b_3^2 - 3245b_3^3\kappa + 2b_3^4 + 15120}{25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464} \\
\eta_2 &= -\frac{2\left(11340b_3\kappa + 366b_3^2 + 1749b_3^3\kappa - 164b_3^4 - 7560\right)}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\eta_3 &= \frac{4\left(21690b_3\kappa + 246b_3^2 - 357b_3^3\kappa + 44b_3^4 + 7560\right)}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)}
\end{aligned} \tag{A.7}$$

- $b_4 \neq 0$

$$\begin{aligned}
\epsilon_1 &= -\frac{3450b_4\kappa - 15095b_4^2 + 1028160}{27288b_4\kappa + 25014b_4^2 - 1410696} \\
\epsilon_2 &= -\frac{3450b_4\kappa + 7645b_4^2 - 514080}{27288b_4\kappa + 25014b_4^2 - 1410696} \\
\eta_1 3 &= \frac{3330b_4\kappa - 65b_4^2 + 7560}{9096b_4\kappa + 8338b_4^2 - 470232}
\end{aligned}$$

$$\eta_2 = \frac{62178 b_4 - 1894 b_4^2 \kappa - 1137 b_4^3 + 3780\kappa}{13644 b_4 \kappa^2 + 12507 b_4^2 \kappa - 705348\kappa}$$

$$\eta_3 = \frac{7410 b_4 \kappa + 565 b_4^2 + 7560}{13644 b_4 \kappa + 12507 b_4^2 - 705348} \tag{A.8}$$

- $b_5 \neq 0$

$$\epsilon_1 = -\frac{3450 b_5 \kappa + 15095 b_5^2 - 1028160}{27288 b_5 \kappa - 25014 b_5^2 + 1410696}$$

$$\epsilon_2 = \frac{3450 b_5 \kappa + 3725 b_5^2 - 257040}{13644 b_5 \kappa - 12507 b_5^2 + 705348}$$

$$\eta_1 = -\frac{60 b_5 \kappa - 65 b_5^2 + 3780}{4548 b_5 \kappa - 4169 b_5^2 + 235116}$$

$$\eta_2 = -\frac{7590 b_5 \kappa - 760 b_5^2 + 3780}{13644 b_5 \kappa - 12507 b_5^2 + 705348}$$

$$\eta_3 = \frac{3593 b_5^2 \kappa - 2274 b_5^3 + 114366 b_5 + 15120\kappa}{25014 b_5^2 \kappa - 27288 b_5 - 1410696\kappa} \tag{A.9}$$

Again we observe that the mixing parameters η_i, that were vanishing in the 4d case, are non-zero in two dimensions.

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